Population (cont'd)

Ex: A population of mice inhabit a field. Every month (month = 30 days), every pair of mice produces a single mouse baby. Every day, local owls eats 15 mice.

Ques: what happens to population over time?

$p(t)$ = population of mice at time t in months.

what is the initial population?
what if $p(0) = 600$ mice?
what if $p(0) = 1000$ mice?

Aside

what if every pair produces 10 mouse babies?

$$p' = \underbrace{5p}_{\#\text{ added by reproduction}} - \overbrace{450}^{\text{killed}}$$

$$p' = 0.5(10)p - 450$$

$$\boxed{p' = 0.5p - 450} \quad \text{or} \quad \boxed{p' - 0.5p = -450}$$

Linear first order

solve to find $p(t)$

$$\boxed{p(t) = 900 + Ce^{0.5t}}$$

general solution

$$\boxed{p = 900 + Ce^{0.5t}}$$

*** Solve to find $p(t)$ ***

STEP 1 a solution to complementary eqn:

$$p' = 0.5p$$

$$\frac{p'}{p} = 0.5$$

$$\ln|p| = 0.5t + k \quad , \text{choose } k=0$$

$$p_1 = e^{0.5t}$$

STEP 2 guess $p = u \cdot p_1$

$$p = u \cdot e^{0.5t}$$

$$p' = 0.5ue^{0.5t} + u'e^{0.5t}$$

STEP 3 substitute into $p' - 0.5p = -450$

$$\cancel{0.5ue^{0.5t}} + u'e^{0.5t} - \cancel{0.5ue^{0.5t}} = -450$$

$$u'e^{0.5t} = -450$$

$$\int u' \, dt = \int -450e^{-0.5t} \, dt$$

$$u = \frac{-450}{-0.5}e^{-0.5t} + C$$

$$u = 900e^{-0.5t} + C$$

STEP 4

$$p = u \cdot p_1$$

$$p = (900e^{-0.5t} + C)e^{0.5t}$$

general solution $\boxed{p = 900 + Ce^{0.5t}}$

① what if $p(0) = 600$ mice?

② what if $p(0) = 1000$ mice?

① find C if $p(0) = 600$

substitute $t=0, p=600$: $600 = 900 + Ce^{0.5 \cdot (0)}$

$$600 = 900 + C \cdot 1$$

$$-900 \quad -900$$

① ANS $\boxed{-300 = C}$

$$\boxed{p(t) = 900 - 300e^{0.5t}}$$

population over time ($p(0) = 600$)

② if $p(0) = 1000$?

sub

$$1000 = 900 + ce^{0.5(0)}$$

$$1000 = 900 + c$$
$$-900 \quad -900$$
$$\boxed{100 = c}$$

$$\boxed{p(t) = 900 + 100e^{0.5t}}$$

what if $c = 0$?

$$p(t) = 900 + ce^{0.5t}$$

set $c = 0$

$$p(t) = 900 + \cancel{0e^{t}}$$

$$\boxed{p(t) = 900}$$

always same population of mice $p = 900$.

$p(0) = 900$

we call this the equilibrium solution

$p' = 0.5(900 - 450$

$p' = 450 - 450 = 0$

TO FIND EQUILIBRIUM of

$$p(t) = 900 + ce^{0.5t}$$

set $p'(t) = 0$, solve for c.

$$p'(t) = 0 + 0.5ce^{0.5t}$$

$$\frac{\cancel{0.5}ce^{0.5t}}{.5e^{0.5t}} = \frac{0}{0.5e^{0.5t}} = 0$$

$\boxed{c = 0}$ ← value for equilibrium solution.

$\boxed{\text{Eq. solution is } p = 900}$

You take a hot cup of coffee outside in the middle of winter. What happens to the temperature T(t) of the coffee over time?

what if outside temp = 50° ? F

what if outside temp = 11° ? F

coffee cools faster!

The change in temperature is related to the difference between the temp of coffee and the temp of environment

"The rate of change of temperature is proportional to the DIFFERENCE between the temperature T of the object and the temperature T_m of the environment (or medium)"

NEWTON'S LAW OF COOLING:
$T' = -k(T - T_m)$
- T(t) is the temperature at time t
- T_m is the temperature of the medium (environment)
- k is a positive quantity, the *temperature decay constant* (depends on surface area of object and various properties of the environment)

$T = T(t)$ = temp. of our object at time t.
T_m = temp. of environment, T_m is constant
K = constant (temp. decay constant)

$$\boxed{T' = -K(T - T_m)}$$

unknown function is
$T(t) =$ ______ ?
the temperature of object at time t.

NEWTON'S LAW OF COOLING
You take a hot cup of coffee outside in the middle of winter. What happens to the temperature T(t) of the coffee over time?

Does it continue to cool forever?
Does it cool more quickly if the temperature outside is 22°F (New York) or -40°F (Alaska)?

"The rate of change of temperature is proportional to the DIFFERENCE between the temperature T of the object and the temperature T_m of the environment (or medium)"

NEWTON'S LAW OF COOLING:
$T' = -k(T - T_m)$
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- k is a positive quantity, the *temperature decay constant* (depends on surface area of object and various properties of the environment)

NOTE: If T_0 is the initial value of T, then the general solution is $T = T_m + (T_0 - T_m)e^{-kt}$

Resource on coffee temperatures: https://driftaway.coffee/temperature/
Example (update): A n extra-hot cup of coffee at 180°F is carried outside in 22°F weather. After 5 minutes, the temperature of the coffee has dropped to 160°F. How long does it take the coffee to reach the perfect temperature of 130°F?

T=22+(158)e^(-0.027068t)
t=14.0559 minutes

Example (OLD): A cup of coffee at 200°F is carried outside in 22°F weather. After 5 minutes, the temperature of the coffee has dropped to 120°F. How long does it take the coffee to reach 75°F?
T=22+178e^(-0.11936t)
t=10.1499 minutes

Example (update): A n extra-hot cup of coffee at 180°F is carried outside in 22°F weather. After 5 minutes, the temperature of the coffee has dropped to 160°F. How long does it take the coffee to reach the perfect temperature of 130°F?

$T_m = 22°$ temp. of environment.

Guess: 10 min?

$T(t)$

$\longrightarrow T = 180$ at $t = 0$, or $T(0) = 180$

$T = 160$ $t = 5$ $T(5) = 160$

STRATEGY ① plug in $T_m = 22$

$$\boxed{T' = -k(T-22)}$$

solve this diff. equation (treat k as constant)

$T(t) =$ __________ (k, C)

substitute $T(0) = 180$

substitute $T(5) = 160$

Solve for k, C.

$\boxtimes$