Today we will use t as our variable

$f(x) = x^2 + x$

$y(x) = x^2 + x$

$\boxed{y(1)}$

Defn: A **Bernoulli Equation** has the form $y' + p(t)y = f(t)y^r$, where r is any real number except 0 or 1.
STEP 1: Let $y_1(t)$ be a solution to the complementary equation $y' + p(t)y = 0$.
STEP 2: Guess a solution of the form $y = uy_1$, where $u(t)$ is some (unknown) function of t.
STEP 3: Substitute into the original equation and separate variables. Then integrate to find u.

Defn. Bernoulli Equation

$$y' + p(t)y = f(t)y^r$$

(r is any real number $r \neq 0$, $r \neq 1$)

$$\boxed{\text{Example 1} \quad \text{Solve } y' - y = ty^2, \quad y(1) = e}$$

Diffy Q

Initial Value (answer will not have "c", but a specific number)

Goal: $y =$ ____

Bernoulli Equation.

STEP 1 complementary equation, single solution y_1

$$y' - y = 0$$
$$+y \quad +y$$
$$\frac{y'}{y} = \frac{y}{y}$$
$$\int \frac{y'}{y} \cancel{dx} = \int 1\,dx$$
$$\ln|y| = x + K$$

solve for y:

$$e^{\ln|y|} = e^{x+K}$$

constant

$$|y| = e^x \cdot e^K$$

constant c

$$y = \pm e^K e^x$$

$$\boxed{y = ce^t} \text{ general soln to complementary equation}$$

You are free to choose any value of c to get a single solution. Choose something nice.

to get a single solution, choose $c = 1$

$$\boxed{y_1 = e^t} \rightarrow \text{End of Step 1} \leftarrow$$

STEP 2 guess $\boxed{y = u \cdot y_1}$

$$y = u \cdot e^t$$

$y' = u \cdot e^t + e^t \cdot u'$

STEP 3 substitute $y' - y = ty^2$,

$ue^t + e^t u' - ue^t = t(u \cdot e^t)^2$

$e^t u' = t(u^2 e^{2t})$

$\frac{\cancel{e^t} u'}{e^t u^2} = \frac{t u^2 e^{2t}}{e^t u^2}$ ← separable - put u's on left, t's on right

$\frac{u'}{u^2} = te^t$

$\int \frac{1}{u^2} \cdot \underbrace{\overset{\frac{du}{dt}}{u'} dt}_{du} = \int te^t \, dt$

$\int u^{-2} du =$

$\int te^t dt$

$p = t \quad dq = e^t dt$

$dp = 1dt \quad q = e^t$

$\frac{u^{-1}}{-1} = \quad = te^t - \int e^t dt$

$= te^t - e^t + C$

Integration by Parts

$\cancel{\int}(pq)' = \int pq' + \int qp'$

$pq = \int pq' + \int qp'$

$- \int qp' \qquad - \int qp'$

$pq - \int qp' = \int pq'$

$\frac{u^{-1}}{-1} = te^t - e^t + C$

$u^{-1} = -te^t + e^t - C$

$\frac{1}{u} = -te^t + e^t - C$

$$\boxed{u = \frac{1}{-te^t + e^t - C}}$$

$y = u \cdot y_1 = \frac{1}{-te^t + e^t - C} \cdot e^t$

$$\boxed{y = \frac{e^t}{-te^t + e^t - C}}$$ General solution

this is a family of solutions.
what is C?

initial condition $y(1) = e$

substitute values

$$e = \frac{e^1}{-\cancel{1e^1} + \cancel{e^1} - C}$$

$$(-c)\, e = \frac{e}{\cancel{-c}} \cdot \cancel{(-c)}$$

$$\frac{-Ce}{e} = \frac{e}{e}$$

$$-C = 1$$

$$C = -1$$

Particular Solution

$$\boxed{y = \frac{e^t}{-te^t + e^t + 1}}$$

a new type of "homogeneous"

Defn a diffy Q is called **homogeneous** if it can be written $y' = f\left(\frac{y}{t}\right)$

STEP 1: In this case, we **always** use $y_1 = t$. MEMORIZE IT!
STEP 2: Guess a solution of the form $y = uy_1$ (so $y = u\boldsymbol{t}$).
STEP 3: Substitute into the original equation, rearranging to use $u = \frac{y}{t}$ when necessary. Then integrate to find u.

Example $y' = \dfrac{y + te^{-\frac{y}{t}}}{t}$

rewrite: $y' = \frac{y}{t} + \frac{te^{-\frac{y}{t}}}{t} = \frac{y}{t} + e^{-\frac{y}{t}}$

Step1 $\boxed{y_1 = t}$ always same for "homogeneous"

Step2 Guess $y = u \cdot y_1 = u \cdot t$ $y = ut$, $\frac{y}{t} = u$

Step3 substitute + solve.
Separate variables, integrate to find u,
$y = u \cdot y_1$

START of Office Hours - finish the problem above

$$y' = \frac{y}{t} + e^{-\frac{y}{t}}$$

STEP 1: $y_1 = t$

STEP 2: Guess $\boxed{y = u \cdot t}$ $\longleftrightarrow$ $\boxed{u = \frac{y}{t}}$

take derivative using product rule.

Step 3: Substitute and solve

$$\boxed{y' = u \cdot 1 + t \cdot u'}$$

$$u + tu' = u + e^{-u}$$
$$-u \qquad\qquad -u$$

separate and integrate

$$\frac{tu'}{te^{-u}} = \frac{e^{-u}}{te^{-u}}$$

$$\frac{u'}{e^{-u}} = \frac{1}{t}$$

$$\int e^{u} \cdot \underbrace{u'\,dt}_{du} = \int \frac{1}{t}\,dt$$

$$e^{u} = \ln|t| + C$$

solve for u:

$$\ln e^{u} = \ln(\ln|t| + C)$$

$$u = \ln(\ln|t| + C)$$

$$y = u \cdot t$$

$$\boxed{y = \ln(\ln|t| + C) \cdot t}$$

General solution

$f = x^2, f' = 2x$

$g = \sin x, g' = \cos x$

$$\frac{d}{dx}(x^2 \cdot \sin x)$$

$$\frac{d}{dx}(f \cdot g) = f \cdot g' + g \cdot f'$$

$$= \boxed{x^2 \cdot \cos x + \sin x \cdot 2x}$$