First Order Linear Nonhomogeneous Diffy Qs.

STRATEGY to solve $y' + p(x)y = f(x)$:
STEP 1: Solve the complementary equation $y' + p(x)y = 0$, let $y_1(x)$ be a solution.
STEP 2: Look for a solution of the form $y = uy_1$. Substitute into the original equation.
STEP 3: Solve for u by isolating u' and integrating.
STEP 4: Substitute $y_1(x)$, $u(x)$ to find the final solution: $y = uy_1$

Ex: Find all solutions:
$$y' + 2y = x^3 e^{-2x}$$

$y' + p(x)y = f(x)$
$p(x) = 2$
$f(x) = x^3 e^{-2x}$
First order
linear
nonhomogeneous
Goal:
$y = \_\_\_\_$

Step 1: $y' + 2y = 0$ complementary equation
$-2y \quad -2y$

$$\frac{y'}{y} = \frac{-2y}{y}$$

$$\int \frac{y'}{y}\,dx = \int -2\,dx$$

$$\ln|y| = -2x + K$$

$$e^{\ln|y|} = e^{-2x+K}$$

$$|y| = e^{-2x+K}$$

$$y = \pm e^{-2x+K}$$

$$y = \pm e^{K} e^{-2x}$$

since $\pm e^{K}$ is a constant, lets call it $c = \pm e^{K}$

$$y = ce^{-2x}$$

$\int \frac{1}{x}\,dx = \ln|x|$

$\int \frac{1}{y} \cdot y'\,dx = \ln|y|$

$y' = \frac{dy}{dx}$

$\int \frac{1}{y} \cdot \frac{dy}{dx} \cdot dx$

$\int \frac{1}{y}\,dy = \ln|y|$

pick a value for c to get a solution $(c = 1)$

$$\boxed{y_1 = e^{-2x}}$$

Step 2: Guess solution has form: $u(x)$

$$y = u \cdot y_1$$

$$\boxed{y = u \cdot e^{-2x}}$$

Can we find u?

substitute into:

$$\boxed{y' + 2y = x^3 e^{-2x}}$$

first find $y' = u \cdot e^{-2x} \cdot (-2) + e^{-2x} \cdot u'$

$$u \cdot e^{-2x}(-2) + e^{-2x} \cdot u' + 2(u \cdot e^{-2x}) = x^3 e^{-2x}$$

$$\cancel{-2ue^{-2x}} + e^{-2x} \cdot u' + \cancel{2ue^{-2x}} = x^3 e^{-2x}$$

$$\frac{\cancel{e^{-2x}} \cdot u'}{\cancel{e^{-2x}}} = \frac{x^3 \cancel{e^{-2x}}}{\cancel{e^{-2x}}}$$

$$u' = x^3$$

$$\int u'\,dx = \int x^3\,dx$$

$$\boxed{u = \frac{x^4}{4} + C}$$

$y = u \cdot e^{-2x}$

substitute:

$$\boxed{y = \left(\frac{x^4}{4} + C\right)e^{-2x}}$$

FINAL ANSWER

Ex: $y' = \frac{-x}{y}$

$+\frac{x}{y} \quad +\frac{x}{y}$

First order

Is it linear?

$y' + p(x)y = f(x)$

$y' + \frac{x}{y} = 0$

$y' + x \cdot \frac{1}{y} = 0$

$y' + x \cdot y^{-1} = 0$

~~$y' + p(x)y = 0$~~

$\cancel{y} \cdot y' = -\frac{x}{\cancel{y}} \cdot \cancel{y}$

$y \cdot y' = -x$

$y' + p(x)y = f(x)$

$p(x) = 1, \; f(x) = -x$

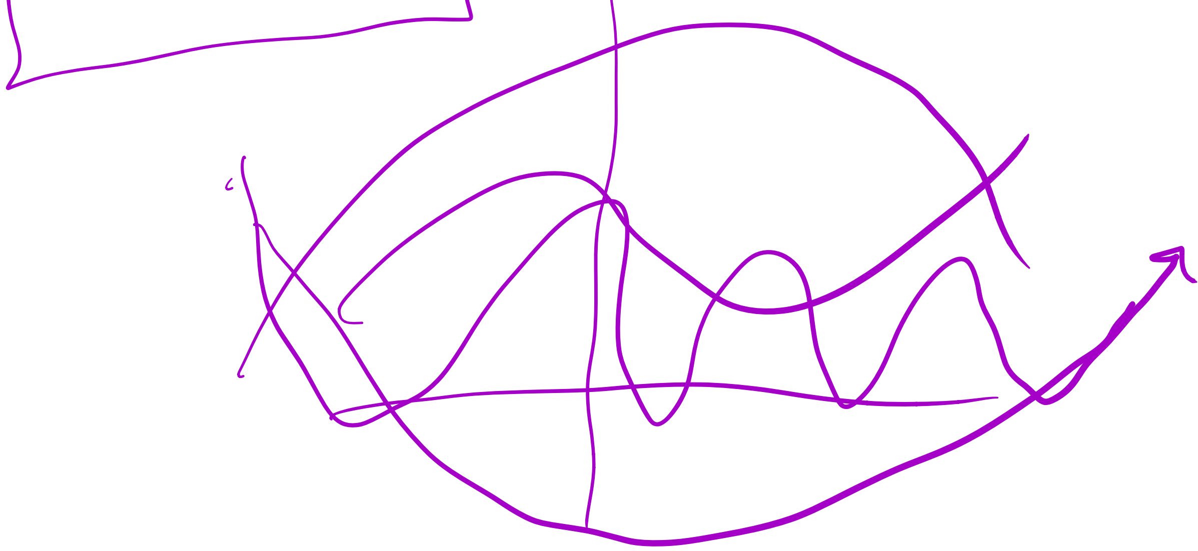
$y' + 1 \cdot y = -x$

don't match

Not First Order Linear.

Its a new type!

Aside think about $\boxed{y' = \frac{-x}{y}}$, Goal: Find $y = \_\_\_$



y' gives us the slope of y.

<u>If</u> our solution $y = \_\_\_\_$ passes through (1, 2) then what is the slope there?

$$\boxed{y' = \frac{-x}{y}}$$

find slope y' at (1, 2)
x y

$$y' = \frac{-x}{y} = \frac{-1}{2}$$

slope at (2, -1)?

$$y' = \frac{-2}{-1} = 2$$

plot slope at a bunch of points to get

a slope field

Solve $y' = -\frac{x}{y}$

Defn: a Diffy Q is called separable if it can be written:

$$h(y) \cdot y' = g(x)$$

Ex: $y \cdot \boxed{y' = \frac{-x}{y}} \cdot y$ → Goal: find $y = $ ____

$g(x) = -x$

$h(y) = y$

$y \cdot y' = -x$

This is separable.

Integrate both sides:

$$\int y \cdot y' \, dx = \int -x \, dx$$

$$\int y \cdot \frac{dy}{dx} \, dx = \int -x \, dx$$

$$\int y \, dy = \int -x \, dx$$

$$\boxed{\frac{y^2}{2} = \frac{-x^2}{2} + C}$$

(*) Implicit Solution (don't have $y=$ by itself)

Find explicit solution by solving for y:

$$y^2 = -x^2 + 2C$$

$$\boxed{y = \pm\sqrt{-x^2 + 2C}}$$

$$\boxed{y = \sqrt{-x^2+2c} \quad \text{or} \quad y = -\sqrt{-x^2+2c}}$$ *

Ques: what solution has $\boxed{y(1)=2}$?
and what interval is it defined on?

$(-x)^2$

sub $x=1, y=2$

$$2 = \sqrt{-(1)^2+2c}$$

$$2^2 = \cancel{\sqrt{-1+2c}}^{\,2}$$

$-x^2 =$

$$2^2 = -1+2c$$

$-1 \cdot x^2$

$$4 = -1+2c$$

$+1 \qquad +1$

$$\frac{5}{2} = \frac{2c}{2}$$

$-2x^2$

$$c = \frac{5}{2}$$

$$y = \sqrt{-x^2+2\cdot\frac{5}{2}}$$

$$\boxed{y = \sqrt{-x^2+5}}$$ *

check: $y' = \frac{-x}{y}$ //

check: $y(1)=2$ //