Using the Laplace Transform to solve differential equations/initial value problems

function	Laplace transform
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, n=positive integer	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$, $s > 0$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$, $s > 0$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$t^n e^{at}$, n=positive integer	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

In this class, we refer to the unknown function as y or $y(t)$.

Function	Laplace Transform
$y(t)$	$Y(s)$
y' or $y'(t)$	$sY(s) - y(0)$
y'' or $y''(t)$	$s^2Y(s) - sy(0) - y'(0)$

Example Use the Laplace Transform to solve:

$$y'' - y' - 2y = 0, \text{ with } y(0) = 1,\ y'(0) = 0$$

Reminder: solving IVPs with the Laplace Transform.
1. take Laplace Transform of both sides of the equation.
2. solve for $Y(s)$
3. simplify if needed.
4. take $\mathcal{L}^{-1}$ of both sides to get y.

$$y'' - y' - 2y = 0$$

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$

$$s^2Y(s) - s\cdot 1 - 0 - (sY(s) - 1) - 2Y(s) = 0 \qquad \square = \square$$

$$s^2Y(s) - s - sY(s) + 1 - 2Y(s) = 0$$

$$+s \quad -1 \qquad +s-1$$

$$s^2Y(s) - sY(s) - 2Y(s) = s - 1$$

$$\frac{Y(s)(s^2 - s - 2)}{s^2-s-2} = \frac{s-1}{s^2-s-2}$$

$$\boxed{Y(s) = \frac{s-1}{s^2-s-2}}$$

factor $s^2 - s - 2 = (s-2)(s+1)$

$$Y(s) = \frac{s-1}{(s-2)(s+1)}$$

partial fractions:

$$\frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

multiply both sides by $(s-2)(s+1)$

$$(s-2)(s+1)\cdot\frac{s-1}{(s-2)(s+1)}=(s-2)(s+1)\left(\frac{A}{s-2}+\frac{B}{s+1}\right)$$

$$s-1=(s+1)A+(s-2)\cdot B$$

$$s-1=As+A+Bs-2B$$

$$s-1=s(A+B)+(A-2B)$$

$$1=A+B$$

$$-1=A-2B$$

$$1-B=A$$

sub $-1=(1-B)-2B$

$$-1=1-B-2B$$

$$-1 \quad -1$$

$$\frac{-2}{-3}=\frac{-3B}{-3}$$

$$\boxed{\frac{2}{3}=B}$$

$$1-\frac{2}{3}=A$$

$$\boxed{\frac{1}{3}=A}$$

$$Y(s)=\frac{\frac{1}{3}}{s-2}+\frac{\frac{2}{3}}{s+1}$$

$$Y(s)=\frac{1}{3}\cdot\frac{1}{s-2}+\frac{2}{3}\cdot\frac{1}{s+1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{3}\,\frac{1}{s-2}+\frac{2}{3}\,\frac{1}{s+1}\right\}$$

$$y(t)=\frac{1}{3}\mathcal{L}^{-1}\Big\{\underbrace{\frac{1}{s-2}}_{a=2}\Big\}+\frac{2}{3}\mathcal{L}^{-1}\Big\{\underbrace{\frac{1}{s+1}}_{a=-1}\Big\}$$

$$y(t)=\frac{1}{3}\cdot e^{2t}+\frac{2}{3}e^{-t}$$

$$\boxed{y(t)=\frac{1}{3}e^{2t}+\frac{2}{3}e^{-t}}$$ ← the particular solution to our initial value problem.

Example Solve $y''+y=\sin 2t,\ y(0)=2,\ y'(0)=1.$

Function	Laplace Transform
$y(t)$	$Y(s)$
y' or $y'(t)$	$sY(s)-y(0)$
y'' or $y''(t)$	$s^2Y(s)-sy(0)-y'(0)$

$$s^2Y(s) - 2s - 1 + Y(s) = \frac{2}{s^2+2^2}$$

$$\quad +2s \quad +1 \qquad\qquad\qquad +2s+1$$

$$s^2Y(s) + Y(s) = \frac{2}{s^2+2^2} + 2s + 1$$

$$\frac{Y(s)(s^2+1)}{s^2+1} = \frac{\frac{2}{s^2+2^2} + 2s + 1}{s^2+1}$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s+1}{s^2+1}\cdot\frac{(s^2+4)}{(s^2+4)}$$

$$= \frac{2}{(s^2+4)(s^2+1)} + \frac{(2s+1)(s^2+4)}{(s^2+4)(s^2+1)}$$

$$= \frac{2}{(s^2+4)(s^2+1)} + \frac{2s^3+8s+s^2+4}{(s^2+4)(s^2+1)}$$

$$Y(s) = \frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)}$$

factor

$s^2+4 = (s+2)\cancel{(s-2)}$

can't be factored - it's irreducible.

s^2+1 doesn't factor (irreducible).

leave them

Use partial fractions

$$\frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

multiply by $(s^2+4)(s^2+1)$ both sides.

$$\cancel{(s^2+4)}\cancel{(s^2+1)}\frac{2s^3+s^2+8s+6}{\cancel{(s^2+4)}\cancel{(s^2+1)}} = (s^2+4)(s^2+1)\left(\frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}\right)$$

$$2s^3+s^2+8s+6 = (s^2+1)(As+B) + (s^2+4)(Cs+D)$$

$$'' \qquad = As^3 + Bs^2 + As + B + Cs^3 + Ds^2 + 4Cs + 4D$$

collect terms by powers of s:

$$2s^3+s^2+8s+6 = s^3(A+C) + s^2(B+D) + s(A+4C) + (B+4D)$$

$$\boxed{\begin{aligned} 2 &= A + C \\ 1 &= B + D \\ 8 &= A + 4C \\ 6 &= B + 4D \end{aligned}}$$

$2 - C = A$

$8 = (2 - C) + 4C$

$8 = 2 - C + 4C$

$-2 \quad -2$

$6 = 3C$

$\boxed{C = 2}$

$A = 2 - 2 = 0$

$\boxed{A = 0}$

$1 - D = B$

$6 = (1 - D) + 4D$

$6 = 1 - D + 4D$

$-1 \quad -1$

$5 = 3D$

$\boxed{D = 5/3}$

$B = 1 - \frac{5}{3}$

$B = \frac{3}{3} - \frac{5}{3} = -\frac{2}{3}$

$\boxed{B = -\frac{2}{3}}$

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+4)(s^2+1)} = \frac{0 \cdot s + \left(-\frac{2}{3}\right)}{s^2+4} + \frac{2s + \frac{5}{3}}{s^2+1}$$

$$Y(s) = \frac{-\frac{2}{3}}{s^2+4} + \frac{2s + \frac{5}{3}}{s^2+1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = -\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{2s + \frac{5}{3}}{s^2+1}\right\}$$

$a = 2$

$$y(t) = -\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{2}{2} \cdot \frac{1}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{5}{3}}{s^2+1}\right\}$$

$$= -\frac{2}{3} \cdot \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t) = -\frac{1}{3}\sin 2t + 2\cos t + \frac{5}{3}\sin t}$$