What is the Laplace Transform?

The Laplace Transform of a function $f(t)$ is another function $\mathcal{L}\{f(t)\} = F(s)$

What is it for/how do we use it?

- We use it to make solving differential equations easier, following this outline:
  1. Start with a differential equation.
  2. Take the Laplace transform of both sides.
     This replaces the differential equation with a much simpler (algebraic) equation.
  3. Solve the algebraic equation.
  4. Simplify the solution.*
     ** requires partial fraction decomposition*
  5. Take the inverse Laplace transform of the solution.
  6. This gives the solution to the original differential equation.
- Yes, but WHAT is the Laplace transform? Ask me about it sometime (go on...)

The following steps can be challenging:

Step 2 * we will learn this today.

step 5 we will learn this Thursday

step 4 next week Tuesday.

Defn If $f(t)$ is a function** defined for $t>0$, then the Laplace Transform of $f(t)$ is:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$f(x)$

$\int_3^5 f(x)\,dx = \#s$

** $f(t)$ must also satisfy:

1. $f(t)$ must be piecewise continuous
2. $f(t)$ "can't be too big" ~ can't be of exponential order:

$$|f(t)| \le Ke^{at} \text{ when } t > M$$

for some constants K, a, M.

Goal: create a table of Laplace Transforms

$f(t)$	$F(s)$
1	$\frac{1}{s}$, $s>0$
t^2	
t^3	
$\cos t$	
$\sin t$	

Example compute the Laplace Transform of the constant function $f(t)=1$.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} \cdot 1 \, dt$$

$$= \int_0^{\infty} e^{-st} \, dt$$

$$= \lim_{n \to \infty} \int_0^{n} e^{-st} \, dt$$

$$= \lim_{n \to \infty} \left[\frac{1}{-s} e^{-st} \right]_0^n$$

$$= \lim_{n \to \infty} \left[\frac{1}{-s} e^{-sn} - \left(\frac{1}{-s} e^{-s \cdot 0} \right) \right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{-se^{sn}} + \frac{1}{s} \right]$$

$$= 0 + \frac{1}{s} = \frac{1}{s} \qquad , s > 0$$

$$\boxed{f(t) = 1 \quad , \quad F(s) = \frac{1}{s} \quad , s > 0}$$

For us, we will use the Table of Laplace Transforms rather than calculating $F(s)$ using the definition.

Table of Laplace Transforms

function	Laplace transform
1	$\frac{1}{s},\ s > 0$
e^{at}	$\frac{1}{s-a},\ s > a$
t^n, n=positive integer	$\frac{n!}{s^{n+1}},\ s > 0$
$\sin at$	$\frac{a}{s^2+a^2},\ s > 0$
$\cos at$	$\frac{s}{s^2+a^2},\ s > 0$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2},\ s > 0$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2},\ s > 0$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2},\ s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2},\ s > a$
$t^n e^{at}$, n=positive integer	$\frac{n!}{(s-a)^{n+1}},\ s > a$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

Ex: Find the Laplace Transform.

1. $f(t) = t^3$ $\quad n=3$

$$F(s) = \frac{3!}{s^{3+1}}, \quad s>0$$

$$\boxed{F(s) = \frac{6}{s^4}, \quad s>0}$$

2. $f(t) = 3t^2 + 7t^8$ $\quad$ by linearity

$$\mathcal{L}\{3t^2 + 7t^8\} = 3\mathcal{L}\{\underset{n=2}{t^2}\} + 7\mathcal{L}\{\underset{n=8}{t^8}\}$$

$$= 3\cdot\frac{2!}{s^3} + 7\cdot\frac{8!}{s^9} \quad s>0$$

$$\boxed{F(s) = \frac{6}{s^3} + \frac{282240}{s^9} \quad s>0}$$

3. $f(t) = \sin 5t + e^{2t}\cos 6t$

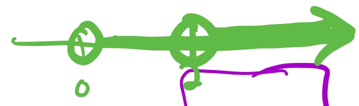
$$\mathcal{L}\{\sin 5t + e^{2t}\cos 6t\} =$$

$$\mathcal{L}\{\sin 5t\} + \mathcal{L}\{e^{2t}\cos 6t\} =$$

$a=5$ $\quad a=2 \quad b=6$

$\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$		$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$

$$\boxed{\frac{5}{s^2+5^2} + \frac{s-2}{(s-2)^2+6^2}, \ s>2}$$



$\boxed{s>0, \ s>2}$

Office Hours

Euler Equations - Trench: Problem 3

(1 point) Library/Rochester/setDiffEQ9Linear2ndOrderHomog/ur_de_9_18.pg

This set is **visible to students.**

Find y as a function of x if

$$x^2y'' - 5xy' - 16y = 0,$$

$y(1) = -1, \;\; y'(1) = -10.$

$y =$ ☐

Euler equation

guess $y = x^r$

indicial equation $r(r-1) - 5r - 16 = 0$

$$r^2 - r - 5r - 16 = 0$$

$$r^2 - 6r - 16 = 0$$

$$(r+2)(r-8) = 0$$

$r = -2 \qquad r = 8$

general solution:

$$\boxed{y = c_1 x^{-2} + c_2 x^8}$$

$y(1) = -1, \quad y'(1) = -10$

$$\boxed{y' = -2c_1 x^{-3} + 8c_2 x^7}$$