Exam Review

Reduction of Order

Ques #2, 7, 8

Review Problems

1. Find the general solution: $y'' - 2y' - 3y = 3e^{2t}$
2. Find the general solution: $y'' + 6y' + 9y = -578\sin 5t$
3. Use the method of reduction of order to find the general solution to $x^2y'' - xy' + y = x$ given that $y_1 = x$ is a solution to the complementary equation.
4. Use the method of reduction of order to find the general solution to $xy'' - (2x+2)y' + (x+2)y = 0$ given that $y_1 = e^x$ is a solution.
5. Given the differential equation $y'' - xy' - y = 0$:
   a. Suppose that $y(x)$ has a Taylor series about $x = 0$,
   $y(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \ldots$
   Substitute into the differential equation and simplify by grouping together terms with

   b. Given the initial conditions $y(0) = 16,\ y'(0) = 15$, find the first five terms of the Taylor series solution $y(x)$.
   c. Use the answer to part b to find an approximation of $y(2)$
6. Given the differential equation $y'' + x^2y = 0$ with initial conditions $y(0) = 1,\ y'(0) = 0$, use the first five terms of the Taylor series about $x = 0$ to find an approximate value of the solution at $x = 1.2$.
7. Suppose y is the solution to a given initial value problem and y is given to you in the form of a MacLaurin series, $y(x) = 11 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{7}{12}x^3 + \frac{51}{24}x^4 + \ldots$.
   a. Find the values of $y(0),\ y'(0),\ y''(0),\ y'''(0),\ y^{(4)}(0)$ (Note that the notation $y^{(4)}(x)$ indicates the fourth derivative of y).
   b. The next term in the MacLaurin series would be a_5x^5. Find the value of the coefficient a_5, given that $y^{(5)}(0) = \frac{3}{7}$ (that is, the fifth derivative of y evaluated at x=0 is $\frac{3}{7}$).
8. Find a particular solution of $x^2y'' + xy' - y = 2x^2 + 2$ given that $y_1 = x$ and $y_2 = \frac{1}{x}$ are solutions of the complementary equation.
9. Find a particular solution of $xy'' + (2-2x)y' + (x-2)y = e^{2x}$ given that $y_1 = e^x$ and $y_2 = \frac{e^x}{x}$ are solutions of the complementary equation.
10. Find the general solution: $y'' - 2y' + y = 14x^{3/2}e^x$

3. Use the method of reduction of order to find the general solution to $x^2y'' - xy' + y = x$ given that $y_1 = x$ is a solution to the complementary equation.

Guess $y = uy_1$

$\boxed{y = u\cdot x}$

$y' = u\cdot 1 + u'\cdot x$

$\boxed{y' = u + u'x}$

$y'' = u' + u'\cdot 1 + u''\cdot x$

$\boxed{y'' = 2u' + u''\cdot x}$

substitute

$x^2(2u' + u''x) - x(u + u'x) + u\cdot x = x$

$\cancel{2}u'x^2 + u''x^3 - \cancel{ux} - \cancel{u'x^2} + \cancel{ux} = x$

$u'x^2 + u''x^3 = x$ ← should give an equation with only u', u'' (not u). Then substitute $w = u'$, $w' = u''$

substitute $\boxed{w = u'}$

$w' = u''$

$$\frac{wx^2 + w'x^3}{x^2} = \frac{x}{x^2}$$ ← First-order linear

$$w + w'x = \frac{1}{x}$$ ← First-order linear.

Solve this equation:

complementary equation:

$$w + w'x = 0$$
$$-w \qquad -w$$

$$\frac{w'x}{wx} = \frac{-w}{wx}$$

$$\frac{w'}{w} = -\frac{1}{x}$$ variables separated, so integrate.

$$\int \frac{w'}{w}\,dx = \int -\frac{1}{x}\,dx$$

$$\ln|w| = -\ln|x| + c$$

$$e^{\ln|w|} = e^{-\ln|x| + c}$$

$$|w| = e^{-\ln|x|} e^{c}$$ c_1

$$|w| = e^{\ln|x^{-1}|} \cdot c_1$$

$$|w| = |x^{-1}|\, c_1$$

$$w = \pm c_1 x^{-1}$$ c_2.

$$\boxed{w = c_2 \cdot \frac{1}{x}}$$ solution to complementary equation.

choose $c_2 = 1$.

$$\boxed{w_1 = \frac{1}{x}}$$

$\frac{1}{x} = x^{-1}$

$\frac{d}{dx}(x^{-1}) = -x^{-2}$

Guess $\boxed{w = u \cdot \frac{1}{x}}$

Find u', substitute.

$$w' = u \cdot (-x^{-2}) + u' \cdot \frac{1}{x} = -ux^{-2} + \frac{u'}{x}$$

$$w + w'x = \frac{1}{x}$$

$$u \cdot \frac{1}{x} + \left(-ux^{-2} + \frac{u'}{x}\right)x = \frac{1}{x}$$

$$\cancel{\frac{u}{x}} - \cancel{ux^{-1}} + u' = \frac{1}{x}$$
$$-\frac{u}{x}$$

$$\int u'\,dx = \int \frac{1}{x}\,dx$$

$$u = \ln|x| + c$$

sub into guess

$$w = u \cdot \frac{1}{x} = (\ln|x| + c) \cdot \frac{1}{x}$$

$$\boxed{w = \frac{1}{x}\ln|x| + \frac{c}{x}}$$

Since $w = u'$,

we have

$$\int u'\,dx = \int \frac{1}{x} \cdot \ln|x| + \frac{c}{x}\,dx$$

$u = \int \frac{1}{x}\ln|x|\,dx + c\int \frac{1}{x}dx$

$= \_\_\_\_\_\_\_\_ + c\ln|x| + D$

$\int \frac{1}{x}\ln|x|\,dx$

Integrate by substitution:

$v = \ln|x|$

$dv = \frac{1}{x}dx$

$= \int v\,dv = \frac{1}{2}v^2$

$= \frac{1}{2}(\ln|x|)^2$

$$\boxed{u = \frac{1}{2}(\ln|x|)^2 + c\ln|x| + D}$$

Since $y = ux$,

$$\boxed{y = \left(\frac{1}{2}(\ln|x|)^2 + c\ln|x| + D\right)x}$$

general solution to the original equation.

7. Suppose y is the solution to a given initial value problem and y is given to you in the form of a MacLaurin series, $y(x) = 11 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{7}{12}x^3 + \frac{51}{24}x^4 + \ldots$.
   a. Find the values of $y(0)$, $y'(0)$, $y''(0)$, $y'''(0)$, $y^{(4)}(0)$ (Note that the notation $y^{(4)}(x)$ indicates the fourth derivative of y).
   b. The next term in the MacLaurin series would be a_5x^5. Find the value of the coefficient a_5, given that $y^{(5)}(0) = \frac{3}{7}$ (that is, the fifth derivative of y evaluated at x=0 is $\frac{3}{7}$).

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

$a_n = \frac{y^{(n)}(0)}{n!}$

$a_0 = \frac{y(0)}{0!}$

$a_1 = \frac{y'(0)}{1!}$

$a_2 = \frac{y''(0)}{2!}$

$a_3 = \frac{y'''(0)}{3!}$

etc...

$a_0 = 11$ so:

$11 = \frac{y(0)}{0!}$

$11 = \frac{y(0)}{1}$

$\boxed{11 = y(0)}$

$a_1 = \frac{1}{2}$

$\frac{1}{2} = \frac{y'(0)}{1!}$

$\frac{1}{2} = \frac{y'(0)}{1}$

$\boxed{\frac{1}{2} = y'(0)}$

$a_2 = \frac{3}{8}$

$\frac{3}{8} = \frac{y''(0)}{2!}$

$2 \cdot \frac{3}{8} = \frac{y''(0)}{2} \cdot 2$

$\boxed{\frac{3}{4} = y''(0)}$

For b)

$$\boxed{a_5 = \frac{y^{(5)}(0)}{5!}}$$

Euler Equations

— equations where series solutions don't work because $P_0(x) \neq 0$.

Defn an Euler Equation has form:

$$ax^2y'' + bxy' + cy = 0 \quad , \quad x > 0$$

To solve: guess $\boxed{y = x^r}$ for some constant r.

find y', y'' and substitute

$$y' = r x^{r-1}$$

$$y'' = r(r-1)x^{r-2}$$

substitute:

$$ax^2 \cdot r(r-1)x^{r-2} + bx \cdot rx^{r-1} + cx^r = 0$$

$$ax^{2+r-2} \cdot r(r-1) + bx^{1+r-1} \cdot r + cx^r = 0$$

$$ax^r \cdot r(r-1) + bx^r \cdot r + c \cdot x^r = 0$$

factor out x^r

$$x^r(ar(r-1) + br + c) = 0$$

either $x^r = 0$ or $ar(r-1) + br + c = 0$

since $x>0$, $x^r \neq 0$ so we must have:

Indicial equation $\boxed{ar(r-1) + br + c = 0}$

solve this quadratic equation for r
get two roots r_1, r_2

THREE CASES

① Two real roots r_1, r_2

basic solutions to original equation: $\boxed{y_1 = x^{r_1},\ y_2 = x^{r_2}}$

② Repeated root $r_1 = r_2$

basic solutions: $\boxed{y_1 = x^{r_1},\quad y_2 = \ln(x)\cdot x^{r_1}}$

③ Complex roots $r_1, r_2 = \lambda \pm \omega i$

lambda omega

basic solutions:

$$\boxed{\begin{aligned} y_1 &= x^{\lambda}\cos(\omega \cdot \ln(x)) \\ y_2 &= x^{\lambda}\sin(\omega\cdot \ln(x)) \end{aligned}}$$

Example Solve $x^2y'' - 5xy' + 9y = 0$, $y(1) = 3$, $y'(1) = 5$

Guess: $y = x^r$

Indicial Equation:

$$1\cdot r(r-1) - 5r + 9 = 0$$

$$r^2 - r - 5r + 9 = 0$$

$$r^2 - 6r + 9 = 0$$

quadratic eq:

$$(r-3)(r-3) = 0$$

$r - 3 = 0$ $\quad$ $r-3=0$

$\boxed{r = 3}$ $\quad$ $\boxed{r=3}$

basic solutions

$$y_1 = x^3, \quad y_2 = \ln(x)\cdot x^3$$

general solution:

$$\boxed{y = c_1 x^3 + c_2 \ln(x)\cdot x^3}$$

Initial conditions: $y(1)=3$, $y'(1)=5$

substitute $y(1)=3$

$$3 = C_1 \cdot 1^3 + \cancel{C_2 \underbrace{\ln(1)}_{=0} \cdot 1^3}$$

$$3 = C_1 +$$

$$\boxed{3 = C_1}$$

Find $y = 3x^3 + C_2 \ln x \cdot x^3$

$$y' = 9x^2 + C_2 \ln x \cdot 3x^2 + C_2 \cdot \frac{1}{x} \cdot x^3$$

$$\boxed{y' = 9x^2 + 3C_2 x^2 \cdot \ln x + C_2 \cdot x^2}$$

substitute $y'(1)=5$

$$5 = 9 \cdot 1^2 + 3C_2 \cdot 1^2 \cdot \ln(1) + C_2 \cdot 1^2$$

$$5 = 9 + C_2$$

$$-9 \quad -9$$

$$\boxed{-4 = C_2}$$

$$\boxed{y = 3x^3 - 4\ln(x) \cdot x^3}$$

Exam Review cont'd:

General form: Second order linear nonhomogeneous differential equation

homogeneous:

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$$

General form of solution:

$$\boxed{y = c_1y_1 + c_2y_2 + y_p}$$

$c_1y_1 + c_2y_2$ is the general solution to the complementary equation

y_p is a particular solution to the original equation