Taylor / MacLaurin Series

Setup: if $f(x)$ is a function and $f(x)$ has derivatives of all orders at $x=0$

the the MacLaurin Series of $f(x)$ is:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

if instead we consider a different point $x=c$, we get the Taylor Series of $f(x)$ at c

$$a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \ldots$$

where the coefficients are given by:

$$a_n = \frac{f^{(n)}(c)}{n!}$$

Amazing fact: for most function, the Taylor Series/MacLaurin series will exactly equal the function.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

ex: $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 -$

$\cos(x) = 1 + 0x - \frac{1}{2}x^2 + 0x^3 + \frac{1}{24}x^4 + 0x^5 - \frac{1}{720}x^6 + \ldots$

Ex 1: Given $y'' - xy = 0$, $y(0) = 3$, $y'(0) = 1$, ($f(0)$, $f'(0)$)
estimate the value of $y(2)$ using the first eight terms of the power series.
up to a_7x^7

STEP 1: guess

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 \ldots$$

what do we do with our guess?
find derivatives y', y'',
substitute into the original equation
try to find values of the coefficients
$a_0, a_1, a_2, a_3, \ldots$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 \ldots$$

Find y', y''

$$y' = 0 + a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + 7a_7x^6 + \ldots$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + 7a_7x^6 + \ldots$$

$$y'' = 0 + 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \ldots$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \ldots$$

substitute

$y'' - xy = 0$ ← original equation

put the rest on the next line.

$$(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \ldots)$$
$$-x(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \ldots) = 0$$

Simplify

$$2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \dots$$
$$-a_0x - a_1x^2 - a_2x^3 - a_3x^4 - a_4x^5 - a_5x^6 - a_6x^7 - a_7x^8 + \dots = 0$$

Group terms according to power of x

$$2a_2 + (6a_3 - a_0)x + (12a_4 - a_1)x^2 + (20a_5 - a_2)x^3 + (30a_6 - a_3)x^4$$
$$+ (42a_7 - a_4)x^5 + \dots = 0$$

equate coefficients on left and right sides

Next:

$2a_2 = 0$ $\qquad a_2 = 0$

$6a_3 - a_0 = 0 \xrightarrow{\text{solve for } a_3} a_3 = \frac{a_0}{6}$

$12a_4 - a_1 = 0 \xrightarrow{\text{solve for } a_4} a_4 = \frac{a_1}{12}$

$20a_5 - a_2 = 0 \xrightarrow{\text{solve for } a_5} a_5 = \frac{a_2}{20}$

$30a_6 - a_3 = 0$ $\qquad a_6 = \frac{a_3}{30}$

$42a_7 - a_4 = 0$ $\qquad a_7 = \frac{a_4}{42}$

$\vdots$ $\qquad \vdots$

Next: determine a_0, a_1 from the initial conditions.

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$$a_0 = \frac{f^{(0)}(0)}{0!} = \frac{f(0)}{1}$$

$$\boxed{a_0 = \frac{3}{1} = 3}$$

$$a_1 = \frac{f^{(1)}(0)}{1!} = \frac{f'(0)}{1}$$

$$\boxed{a_1 = \frac{1}{1} = 1}$$

$a_2 = 0$

$a_3 = \frac{a_0}{6} = \frac{3}{6} = \frac{1}{2}$

$a_4 = \frac{a_1}{12} = \frac{1}{12}$

$a_5 = \frac{a_2}{20} = \frac{0}{20} = 0$

$a_6 = \frac{a_3}{30} = \frac{\frac{1}{2}}{30} = \frac{1}{60}$

$a_7 = \frac{a_4}{42} = \frac{\frac{1}{12}}{42} = \frac{1}{504}$

put the coefficients back into our guess

$$y = 3 + x + 0\cdot x^2 + \frac{1}{2}x^3 + \frac{1}{12}x^4 + 0\cdot x^5 + \frac{1}{60}x^6 + \frac{1}{504}x^7 + \dots$$

First eight terms:

$$y \approx 3 + x + \frac{1}{2}x^3 + \frac{1}{12}x^4 + \frac{1}{60}x^6 + \frac{1}{504}x^7$$

approx $y(2)$ using first eight terms,

plug in $x=2$ into

$$y(2) \approx 3 + 2 + \frac{1}{2}\cdot 2^3 + \frac{1}{12} 2^4 + \frac{1}{60}\cdot 2^6 + \frac{1}{504}\cdot 2^7 = \boxed{11.654}$$

$$\boxed{y(2) \approx 11.654}$$