Variation of Parameters

what is it used for?

- Used to find a particular solution y_p to $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$

  second-order linear equation.

- To use this method, we must know the complete general solution to the complementary equation $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$,

  solution $y_h = c_1y_1 + c_2y_2$

- Once we find a particular solution y_p, the general solution to the original equation is $y = c_1y_1 + c_2y_2 + y_p$

- other requirement: $P_0(x)$ is not $= 0$ on any interval of interest.

Beware: much of this method will look familiar, but there will be some unexpected parts.
("extra things to remember")

IDEA: $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$.

If the complementary equation has general solution $y_h = c_1y_1 + c_2y_2$, then we guess a solution to the original equation of the form: $\boxed{y_p = u_1 y_1 + u_2 y_2}$, where u_1 and u_2 are unknown functions of x.

next: find y_p', y_p''.

* BEWARE: we will impose a condition to simplify y_p' (before taking y_p'')

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

Impose the following condition:

$$\boxed{u_1' y_1 + u_2' y_2 = 0}$$

* Important equation #1.

$$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{=0}$$

$$\boxed{y_p' = u_1 y_1' + u_2 y_2'}$$

$$\boxed{y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'}$$

Substitute y_p, y_p', y_p'' into $P_0(x) y'' + P_1(x) y' + P_2(x) y = F(x)$

$$P_0(x)\left(u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'\right) + P_1(x)\left(u_1 y_1' + u_2 y_2'\right) +$$

$$P_2(x)\left(u_1 y_1 + u_2 y_2\right) = F(x)$$

$$P_0(x)u_1y_1'' + P_0(x)u_1'y_1' + P_0(x)u_2y_2'' + P_0(x)u_2'y_2' + P_1(x)u_1y_1' + P_1(x)u_2y_2' + P_2(x)u_1y_1 + P_2(x)u_2y_2 = F(x)$$

Group all terms with u_1, same for u_2:

$$u_1\left(P_0(x)y_1'' + P_1(x)y_1' + P_2(x)y_1\right) + u_2\left(P_0(x)y_2'' + P_1(x)y_2' + P_2(x)y_2\right)$$

$$+ P_0(x)u_1'y_1' + P_0(x)u_2'y_2' = F(x)$$

Where did y_1 and y_2 come from in this problem?

Since y_1, y_2 are solutions to the complementary equation, these parentheses must $= 0$

$$P_0(x)u_1'y_1' + P_0(x)u_2'y_2' = F(x)$$

divide both sides by $P_0(x)$:

$$\boxed{u_1'y_1' + u_2'y_2' = \frac{F(x)}{P_0(x)}}$$

* IMPORTANT EQUATION 2

VARIATION OF PARAMETERS:

TO SOLVE: $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$, given y_1, y_2 (independent) solutions to the complementary equation:

1. GUESS A PARTICULAR SOLUTION: $y_p = u_1y_1 + u_2y_2$
2. WRITE DOWN THE SYSTEM OF EQUATIONS:

$$u_1'y_1 + u_2'y_2 = 0$$
$$u_1'y_1' + u_2'y_2' = \frac{F}{P_0}$$

3. Solve the system of equations for u_1', u_2'
4. Integrate to find u_1, u_2 (let constants of integration =0)
5. Substitute into y_p
6. The general solution is $y = y_p + c_1y_1 + c_2y_2$

Example 1: Find the general solution $x^2y'' - 2xy' + 2y = x^{9/2}$, given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation $x^2y'' - 2xy' + 2y = 0$.

$y_1' = 1$

$y_2' = 2x$

STEP1: Guess $y_p = u_1 x + u_2 \cdot x^2$

STEP2: $u_1'x + u_2'x^2 = 0$

$$u_1' \cdot 1 + u_2' \cdot 2x = \frac{x^{9/2}}{x^2}$$

$$u_1' + u_2' 2x = x^{\frac{9}{2}-2}$$

$$\boxed{\begin{aligned} u_1' + 2xu_2' &= x^{\frac{5}{2}} \\ xu_1' + x^2u_2' &= 0 \end{aligned}}$$

solve this system of equations for u_1', u_2'.

solve for u_1', u_2'

$$u_1' = x^{5/2} - 2xu_2'$$

substitute:

$$x(x^{5/2} - 2xu_2') + x^2u_2' = 0$$

$$x^{7/2} \underbrace{- 2x^2u_2' + x^2u_2'}_{\text{like terms}} = 0$$

$$x^{7/2} - x^2u_2' = 0$$

$$-x^{7/2} \qquad\qquad -x^{7/2}$$

$$\frac{-x^2u_2'}{-x^2} = \frac{-x^{7/2}}{-x^2}$$

$$u_2' = x^{7/2-2}$$

$$\boxed{u_2' = x^{3/2}}$$

substitute u_2' into: $u_1' = x^{5/2} - 2xu_2'$

$$u_1' = x^{5/2} - 2x\left(x^{3/2}\right)$$

$$u_1' = x^{5/2} - 2x^{5/2}$$

$$\boxed{u_1' = -x^{5/2}}$$

STEP4: Integrate u_1', u_2'

$$\int u_1'\,dx = \int -x^{5/2}\,dx$$

$$c = 0$$

$$u_1 = \frac{-x^{7/2}}{\frac{7}{2}} + C = \boxed{-\frac{2}{7}x^{7/2} = u_1}$$

$$\int u_2'\,dx = \int x^{3/2}\,dx$$

$$u_2 = \frac{x^{5/2}}{\frac{5}{2}} + c \quad , \quad c=0 .$$

$$\boxed{u_2 = \frac{2}{5}x^{5/2}}$$

STEP 5: $y_p = u_1 y_1 + u_2 y_2$

substitute: $y_p = -\frac{2}{7}x^{7/2} \cdot x + \frac{2}{5}x^{5/2} \cdot x^2$

$$y_p = -\frac{2}{7}x^{9/2} + \frac{2}{5}x^{9/2}$$

$$y_p = \left(-\frac{2}{7} + \frac{2}{5}\right)x^{9/2}$$

$$= \left(-\frac{10}{35} + \frac{14}{35}\right)x^{9/2}$$

$$\boxed{y_p = \frac{4}{35}x^{9/2}}$$

particular solution to original eq.

general solution to original:

$$y = c_1 y_1 + c_2 y_2 + y_p$$

$$\boxed{y = c_1 x + c_2 x^2 + \frac{4}{35}x^{9/2}}$$

Try it:

Example 2: Find the general solution: $y'' + 3y' + 2y = \frac{1}{1+e^{x}}$

HINT: First find y_h, the general solution to the complementary equation.

ANS: $y_h = c_1e^{-x} + c_2e^{-2x}$, $y_p = (e^{-x} + e^{-2x})\ln(1 + e^{x})$