last time:

second order linear constant coefficient differential equations - nonhomogeneous:

$$ay'' + by' + cy = f(x)$$

Today: second order linear differential equations - homogeneous and nonhomogeneous

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$$

or

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

Reduction of order allows us to solve these provided we know a single solution y_1 to the complementary equation.

STEP 1: Guess a solution to the original equation of the form $y = u \cdot y_1$.
Find y', y'' and substitute into the original equation. Simplify.

STEP 2: The result will have u'' and u', but not u (it cancels).
Substitute $w = u'$, $w' = u''$.
Solve this first-order equation for w.

STEP 3 Now replace w with u', integrate to find u.

STEP 4 General solution is $y = u \cdot y_1$

Ex: Find the general solution to $x^2y''-3xy'+3y=0$, given that $y_1=x$ is a solution.

STEP 1: Guess $\boxed{y=u\cdot x}$

$$y'=u'\cdot x+u\cdot 1=\boxed{u'x+u}$$

$$y''=u''x+u'\cdot 1+u'$$

$$=u''x+u'+u'=\boxed{u''x+2u'}$$

$$x^2(u''x+2u')-3x(u'x+u)+3ux=0$$

$$u''x^3+\underline{2u'x^2}-\underline{3u'x^2}\cancel{-3ux}\cancel{+3ux}=0$$

$$u''x^3-u'x^2=0$$

STEP 2: substitute $w=u'$, $w'=u''$

$$w'x^3-wx^2=0$$

Linear First

$+wx^1$ $\quad +wx^2$

separable.

$$\frac{w'x^3}{w} = \frac{wx^2}{w}$$

$$\frac{\frac{w'}{w}x^3}{x^3} = \frac{x^2}{x^3}$$

$$\int \frac{w'}{w}\,dx = \int \frac{1}{x}\,dx$$

variables are now separated.

$$\ln|w| = \ln|x| + C$$

$$e^{\ln|w|} = e^{\ln|x| + C}$$

$$|w| = e^{\ln|x|} \cdot e^{C} \quad C_1$$

$$|w| = |x| \cdot C_1 \quad C_2$$

$$w = \pm C_1 x$$

$$\boxed{w = C_2 x} \quad \text{or} \quad \underline{w = \pm e^{c} x}$$

STEP 3

$$\int u' \, dx = \int C_2 x \, dx$$

$$u = C_2 \frac{x^2}{2} + C_3$$

STEP 4

$$y = u \cdot y_1$$

$$y = \left(\frac{C_2}{2} x^2 + C_3\right) \cdot x$$

$$y = \underbrace{\frac{C_2}{2}}_{C_4} x^3 + C_3 x$$

$$\boxed{y = C_4 x^3 + C_3 x}$$

general solution to original equation.

Ex: Find the general solution to

$$xy'' - (2x+1)y' + (x+1)y = x^2,$$

given that $y_1 = e^x$ is a solution to the complementary equation.

Guess: $y = u \cdot y_1$

STEP 1: guess $\boxed{y = u \cdot e^x}$

$$\boxed{y' = u'e^x + u \cdot e^x}$$

$$y'' = u''e^x + u'e^x + u'e^x + u \cdot e^x$$

$$\boxed{y'' = u''e^x + 2u'e^x + ue^x}$$

substitute

$$x(u''e^x + 2u'e^x + ue^x) - (2x+1)(u'e^x + ue^x) + (x+1)ue^x = x^2$$

$$u''xe^x + 2u'xe^x + uxe^x - (2u'xe^x + 2uxe^x + u'e^x + ue^x) + uxe^x + ue^x = x^2$$

$$u''xe^x + \cancel{2u'xe^x} + \cancel{uxe^x} - \cancel{2u'xe^x} - \cancel{2uxe^x} - u'e^x - \cancel{ue^x} + \cancel{uxe^x} + \cancel{ue^x} = x^2$$

$$u''xe^x - u'e^x = x^2$$

STEP 2 sub $w = u'$, $w' = u''$

$$\frac{w'xe^x}{xe^x} - \frac{we^x}{xe^x} = \frac{x^2}{xe^x}$$

$$\boxed{w' - \frac{1}{x} \cdot w = \frac{x}{e^x}}$$

First-order linear nonhomogeneous.

Hint: to solve these

1. solve complementary equation y_c

2. guess $y = v \cdot y_c$

find y' substitute

simplify

3 integrate to find v

4. then $y = v \cdot y_c$

this y is our $\underline{\underline{\underline{y}}}$ = ____

Continue with steps 2,3,4