Second order linear differential equations with constant coefficients – nonhomogeneous

Recall: the equations have the form:

$$ay'' + by' + cy = f(x)$$

solutions look like:

$$y = \underbrace{c_1 y_1 + c_2 y_2}_{\text{general solution to complementary equation}} + y_p$$

a single solution to the original equation.

TO SOLVE

STEP 1: Solve complementary equation

STEP 2: "Method of undetermined coefficients"

- first, use the form of $f(x)$, together with the solutions y_1, y_2 to the complementary, to make a guess of $y_p = \underline{\text{Guess}}$.

- find y_p', y_p'', plug in to original equation to find constants.

Recall example (last time):

Ex: $y'' + 9y' = 36$

STEP 1: we found general solution to complementary was: $C_1 \sin 3x + C_2 \cos 3x$

$f(x) = 36$ ← a constant.

STEP 2: guess $y_p = A$ ← a constant.

* Ques: is y_p one of the solutions y_1, y_2 to the complementary equation?

ANS: No.

we keep the guess $y_p = A$.

Recall (last time)

Ex: $y'' - 3y' - 4y = 3e^{2x}$

STEP 1 comp. equation solution $C_1 e^{4x} + C_2 e^{-x}$

STEP 2 $f(x) = 3e^{2x}$

Guess: $y_p = Ae^{2x}$

Ques: is $y_p =$ a solution to complementary?

No. so keep our guess:

$$y_p = Ae^{2x}$$

Ex: $y'' - 7y' + 12y = 5e^{4x}$

STEP 1 solve complementary:

$$y'' - 7y' + 12y = 0$$

guess: $y = e^{rt}$

characteristic eq:

$$r^2 - 7r + 12 = 0$$

$$(r-4)(r-3) = 0$$

$r = 4$ $\qquad$ $r = 3$

Two real roots,

general solution to complementary eq:

$$\boxed{c_1 e^{4x} + c_2 e^{3x}}$$

STEP 2: $f(x) = 5e^{4x}$

Guess: $y_p = Ae^{4x}$

WARNING: e^{4x} is a solution

to the complementary equation,
If we use it, we will end
with 0 on the left side
of our equation instead of $5e^{4x}$.

Revised guess: $\boxed{y_p = Axe^{4x}}$

multiply our
guess by x.

find y_p', y_p'', substitute, solve for A.

$y_p' = Axe^{4x} \cdot 4 + A \cdot 1 \cdot e^{4x}$

$\boxed{y_p' = 4Axe^{4x} + Ae^{4x}}$

$y_p'' = 4Axe^{4x} \cdot 4 + 4A \cdot 1 \cdot e^{4x} + Ae^{4x} \cdot 4$

$= 16Axe^{4x} + 4Ae^{4x} + 4Ae^{4x}$

$\boxed{y_p'' = 16Axe^{4x} + 8Ae^{4x}}$

substitute into $y'' - 7y' + 12y = 5e^{4x}$

$(16Axe^{4x} + 8Ae^{4x}) - 7(4Axe^{4x} + Ae^{4x}) + 12Axe^{4x} = 5e^{4x}$

$$16\cancel{Axe^{4x}} + 8Ae^{4x} - \cancel{28Axe^{4x}} - 7Ae^{4x} + \cancel{12Axe^{4x}} = 5e^{4x}$$

$$Ae^{4x} = 5e^{4x}$$

$$\boxed{A = 5}$$

$$\boxed{y_p = 5xe^{4x}}$$

General solution to original equation:

$$y = c_1 y_1 + c_2 y_2 + y_p$$

$$\boxed{y = c_1 e^{4x} + c_2 e^{3x} + 5xe^{4x}}$$

How do we make our guess of y_p in step 2?

if $f(x)$ is ...	then guess $y_p = \ldots$
a constant C.	a constant A.
e^{rx}	Ae^{rx}
either or both: $\sin x$, $\cos x$	$A\sin x + B\cos x$
a polynomial e.g. $x^3 + 2x$	$Ax^3 + Bx^2 + Cx + D$
$x^2 - 1$	$Ax^2 + Bx + C$
$e^{rx}\sin ax$,	$Ae^{rx}\sin ax + Be^{rx}\cos ax$

$e^{rx}\cos ax$

(*) if $f(x)$ is a solution to the complementary equation, then multiply your guess by x and proceed

Ex: $y'' - 9y' + 14y = 212\sin(2x)$

STEP 1: Solve complementary eq:

$$y'' - 9y' + 14y = 0$$

guess $y = e^{rt}$

$$r^2 - 9r + 14 = 0$$

$$(r-2)(r-7) = 0$$

$r = 2$ $\quad$ $r = 7$

general solution to complementary eq:

$$\boxed{C_1 e^{2x} + C_2 e^{7x}}$$

STEP 2: $f(x) = 212\sin(2x)$

guess: $\boxed{y_p = A\sin 2x + B\cos 2x}$

$$y_p' = A(\cos 2x)\cdot 2 + B(-\sin 2x)\cdot 2$$

$$\boxed{y_p' = 2A\cos 2x - 2B\sin 2x}$$

$$y_p'' = 2A(-\sin 2x)\cdot 2 - 2B(\cos 2x)\cdot 2$$

$$\boxed{y_p'' = -4A\sin 2x - 4B\cos 2x}$$

substitute: $y'' - 9y' + 14y = 212\sin(2x)$

$$-4A\sin 2x - 4B\cos 2x - 9(2A\cos 2x - 2B\sin 2x) + 14(A\sin 2x + B\cos 2x) = 212\sin(2x)$$

$$-4A\sin 2x - 4B\cos 2x - 18A\cos 2x + 18B\sin 2x + 14A\sin 2x + 14B\cos 2x = 212\sin 2x$$

$$(-4A + 18B + 14A)\sin 2x + (-4B - 18A + 14B)\cos 2x = 212\sin 2x$$

$$(10A + 18B)\sin 2x + (-18A + 10B)\cos 2x = 212\sin 2x$$

remember, our goal is to find the values of the constants A and B such that the left side equals the right side.

$10A + 18B = 212$ ← equate coefficients of $\sin 2x$

$-18A + 10B = 0$ ← since $\cos 2x$ doesn't appear on the left, the coefficient should be 0.

solve by substitution method, elimination method, matrices.

$$10A + 18B = 212$$

$$-18A + 10B = 0 \longrightarrow -18A + 10B = 0 \text{ solve for } B$$

$$\frac{10B}{10} = \frac{18A}{10}$$

$$B = \frac{18}{10}A$$

sub into first eq: ← $B = \frac{9}{5}A$

$$10A + 18\left(\frac{9}{5}A\right) = 212$$

$$10A + \frac{162}{5}A = 212$$

$$42.4A = 212$$

$$A = \frac{212}{42.4} = \boxed{5}$$

$$B = \frac{9}{5}A = \frac{9}{5}\cdot 5 = \boxed{9 = B}$$

$$\boxed{y_p = 5\sin 2x + 9\cos 2x}$$

general solution to original eq:

$$y = c_1 y_1 + c_2 y_2 + y_p$$

$$\boxed{y = c_1 e^{2x} + c_2 e^{7x} + 5\sin 2x + 9\cos 2x}$$

characteristic polynomial:

$r^2 - 9r + 14$

roots:

2, 7

y_1, y_2:

e^{2x}, e^{7x}