Exam 2 review

7. A bottle of cold water at 34°F is carried into a warm 85°F room. After 30 seconds, the temperature of the water is 36.2°F.
   a. Assume the time t is measured in seconds, find a formula for the temperature of the water $T(t)$ at time t.

   b. What will the temperature be after 10 minutes?
   c. How long does it take the water to reach 75°F?

a) Newton's Law of Cooling: $T' = -k(T - T_m)$

$T_m = 85$

$T(0) = 34°F$

$T(30) = 36.2°F$

$\frac{T'}{T-85} = \frac{-k(T-85)}{T-85}$ ← separable

$$\frac{T'}{T-85} = -k$$

$$\int \frac{1}{T-85} \cdot T' \, dt = \int -k \, dt$$

$$\int \frac{1}{T-85} \cdot \frac{dT}{dt} \cdot dt$$

$$\ln|T-85| = -kt + C$$

$$e^{\ln|T-85|} = e^{-kt+C}$$

$$|T-85| = e^{-kt}\underbrace{e^{C}}_{C_1}$$

$$T-85 = \underbrace{\pm C_1}_{C_2} e^{-kt}$$

$$T-85 = C_2 e^{-kt}$$

$$+85 \qquad +85$$

$$\boxed{T = C_2 e^{-kt} + 85}$$ general solution

plug in initial conditions into general solution to find C_2, k.

$$T(0) = 34^\circ F$$

$$T(30) = 36.2^\circ F$$

$T(0) = 34$

$$34 = C_2 e^{-k\cdot 0} + 85$$

$$34 = C_2 + 85$$

$$-85 \qquad -85$$

$$\boxed{C_2 = -51}$$

$T(30) = 36.2$

$$36.2 = -51 e^{-k\cdot 30} + 85$$

$$-85 \qquad -85$$

$$\frac{-48.8}{-51} = \frac{-51 e^{-30k}}{-51}$$

$$0.95686275 = e^{-30k}$$

$$\frac{\ln(0.95686275)}{-30} = \frac{-30k}{-30}$$

$$\boxed{k = .001469844}$$

$$\boxed{T = -51e^{-.001469844t} + 85}$$

$T(t)$

solution part a)

b. What will the temperature be after 10 minutes?

c. How long does it take the water to reach 75°F?

b) $t = 10\text{min} \cdot 60\,\text{sec}/\text{min} = 600\,\text{sec}$

$$T(600) = -51e^{-.001469844(600)} + 85$$

$$= 63.88636$$

after 10 min, the temperature will be about $63.89°F$

c) plug in $T = 75$, find t.

$$75 = -51e^{-.001469844t} + 85$$

$$t = 1108.4 \text{ sec}.$$

it will take about 1108 seconds (18.5 minutes) to reach 75° $\square$

Second order linear differential equations with constant coefficients - <u>nonhomogeneous</u>.

<u>Theorem</u> Given a second order linear constant coefficient nonhomogeneous equation:

$$ay'' + by' + cy = f(x)$$

the general solution will have the form:

$$y = c_1 y_1 + c_2 y_2 + y_p$$

where $c_1 y_1 + c_2 y_2$ is the general solution to the complementary equation $ay'' + by' + cy = 0$

and y_p is any particular solution to the original equation.

To solve such an equation, two parts to complete:

<u>Part 1</u>: look at complementary equation and find general solution.

<u>Part 2</u>: to find y_p, we will <u>make a guess</u> and then plug in to find constants

"Method of undetermined coefficients"

Ex 1 solve $y'' + 9y = 36$

Part 1 complementary equation

$$y'' + 9y = 0$$

characteristic equation: guess $y = e^{rx}$

$$r^2 + 0r + 9 = 0$$

$$r^2 + 9 = 0$$

$$-9 \quad -9$$

$$r^2 = -9$$

$$r = \pm\sqrt{-9}$$

$$r = \pm 3i$$

$(r-3)(r+3)$

complex roots: $r = \lambda \pm wi$

general solution: $y = e^{\lambda t}(A \sin wx + B \cos wx)$

$\lambda = 0, \quad w = 3$

general solution to complementary equation is

$$y = e^{0 \cdot t}(A \sin 3x + B \cos 3x)$$

$$\boxed{y_c = A\sin 3x + B\cos 3x}$$

Part 2: $y'' + 9y = 36$

guess $\boxed{y_p = C}$

guess a constant b/c right hand side is $=36$, a constant

find y_p', y_p''

$$\boxed{\begin{aligned} y_p' &= 0 \\ y_p'' &= 0 \end{aligned}}$$

substitute:

$$0 + 9 \cdot C = 36$$

$$\frac{9c}{9} = \frac{36}{9}$$

$$\boxed{c = 4}$$

$$\boxed{y_p = 4}$$

General solution:

$$y = \underbrace{c_1 y_1 + c_2 y_2}\ + y_p$$

$$\boxed{y = A\sin 3x + B\cos 3x + 4}$$

general solution to original equation.

Ex2: solve $y'' - 3y' - 4y = 3e^{2x}$, $y(0) = \frac{17}{2}$
$y'(0) = 10$

Part 1 Complementary:

$$y'' - 3y' - 4y = 0$$

guess $y = e^{rt}$

characteristic: $r^2 - 3r - 4 = 0$

$$(r-4)(r+1) = 0$$

$r - 4 = 0$ $\quad\quad$ $r + 1 = 0$

$\boxed{r = 4}$

$r=-1$

$$\boxed{y_c = Ae^{4x} + Be^{-x}}$$

general soln to complementary eq.

Part 2 $\quad y'' - 3y' - 4y = 3e^{2x}$

guess a solution: $\boxed{y_p = Ce^{2x}}$

Find y', y''

substitute y, y', y'' into original equ., solve for C.

$$y_p' = 2Ce^{2x}$$

$$y_p'' = 4Ce^{2x}$$

sub:

$$4Ce^{2x} - 3\cdot 2Ce^{2x} - 4Ce^{2x} = 3e^{2x}$$

solve for C:

Keep going!

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where does the characteristic equation come from?

Ex: $ay'' + by' + cy = 0$

Guess $y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

substitute:

$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$

factor e^{rx}

$(ar^2 + br + c)e^{rx} = 0$

either $\boxed{ar^2 + br + c = 0}$ or $\cancel{e^{rx} = 0}$

characteristic eq.

$ar^2 + br + c = 0$ characteristic eq.

5. $25y'' - 60y' + 36y = 0$, $y(0) = 5/3$, $y'(0) = 3/2$

guess $y = e^{rx}$

characteristic eq: $25r^2 - 60r + 36 = 0$

$$r = \frac{-(-60) \pm \sqrt{(-60)^2 - 4 \cdot 25 \cdot 36}}{2 \cdot 25}$$

$$r = \frac{60 \pm \sqrt{\cancel{3600 - 3600}}}{50}$$

$$r = \frac{60}{50} = \frac{6}{5}$$ repeated root.

$y = e^{\frac{6}{5}x}$

$$\boxed{y = C_1 x e^{\frac{6}{5}x} + C_2 e^{\frac{6}{5}x}}$$ general solution.

$y(0) = 5/3$, $y'(0) = 3/2$

$$y(0)=\frac{5}{3}$$

$$\frac{5}{3}=\underline{c_1\cdot 0\cdot e^{\frac{6}{5}\cdot 0}}+c_2e^{\frac{6}{5}\cdot 0}$$

$$\frac{5}{3}=c_2e^0$$

$$\boxed{\frac{5}{3}=c_2}$$

$$y=c_1xe^{\frac{6}{5}x}+\frac{5}{3}e^{\frac{6}{5}x}$$

$$y'=\underline{c_1\cdot 1\cdot e^{\frac{6}{5}x}}+\underline{c_1xe^{\frac{6}{5}x}\cdot\left(\frac{6}{5}\right)}+\underline{\frac{5}{3}\cdot\frac{6}{5}e^{\frac{6}{5}x}}$$

$$\boxed{y'=c_1e^{\frac{6}{5}x}+c_1\frac{6}{5}xe^{\frac{6}{5}x}+2e^{\frac{6}{5}x}}$$

plug in $y'(0)=\frac{3}{2}$

$$\frac{3}{2}=c_1e^{\frac{6}{5}\cdot 0}+c_1\cdot\frac{6}{5}\cdot 0e^{\frac{6}{5}\cdot 0}+2e^{\frac{6}{5}\cdot 0}$$

$$\frac{3}{2}=c_1+2$$

$-2 \qquad -2$

$\boxed{-\frac{1}{2} = c_1}$

$$\boxed{y = -\frac{1}{2}xe^{\frac{6}{5}x} + \frac{5}{3}e^{\frac{6}{5}x}}$$ ANS

$r = 2 \pm 9i$

$r_1 = 2 + 9i$

$r_2 = 2 - 9i$

general:

e

$$y = c_3 e^{2t}\cos 9t + c_4 e^{2t}\sin 9t$$

Ex: $10y'' + y' + 10y = 0$

$10r^2 + r + 10 = 0$

solve.

$r = 7 \pm 3i$

general solution:

$y = c_1 e^{7t} \cos 3t + c_2 e^{7t} \sin 3t$