Second order linear homogeneous differential equations with constant coefficients.

$$ay'' + by' + cy = 0$$

STEP 1: Guess a solution of form $y = e^{rx}$

STEP 2: Find y', y'', substitute, simplify.

This always leads to characteristic equation

$$\boxed{ar^2 + br + c = 0}$$

← solve by factoring, or use quadratic formula $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

STEP 3: if the characteristic equation has two real roots r_1 and r_2, the general solution to the original equation will be $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Ques: Does every quadratic equation have two real roots?

- Could have imaginary/complex roots
- Can have a single repeated root.

Single Repeated root:

ex: $y'' + 6y' + 9y = 0$

step 1: guess $y = e^{rx}$

step 2: characteristic equation:

$$r^2 + 6r + 9 = 0$$

$$(r+3)(r+3) = 0$$

$r + 3 = 0$ $r + 3 = 0$

$r = -3$ $r = -3$ repeated root.

imaginary: ai a is real, $i = \sqrt{-1}$

complex: $ai + b$ a, b real, $i = \sqrt{-1}$

try solutions $y = e^{r_1 x}$ and $y = e^{r_1 x}$

$y = e^{-3x}$ ← same! → $y = e^{-3x}$

general: $\boxed{y = c_1 e^{-3x} + c_2 e^{-3x}}$

$y = (c_1 + c_2) e^{-3x}$

$\boxed{y = c_3 e^{-3x}}$ this is only one basic solution. We need two.

Guess the other solution has form

$y = u \cdot y_1$

$\boxed{y = u \cdot e^{-3x}}$

take deriv's substitute into original eq.

$y' = u' \cdot e^{-3x} + u \cdot (-3) e^{-3x}$

$\boxed{y' = u' e^{-3x} - 3u e^{-3x}}$

product rule twice:

$y'' = u'' \cdot e^{-3x} + u'(-3)e^{-3x} + (-3u' e^{-3x}) + (-3u)(-3)e^{-3x}$

$y'' = u'' e^{-3x} \underbrace{-3u' e^{-3x} - 3u' e^{-3x}} + 9u e^{-3x}$

$\boxed{y'' = u'' e^{-3x} - 6u' e^{-3x} + 9u e^{-3x}}$

Substitute into $y'' + 6y' + 9y = 0$

$(u'' e^{-3x} - 6u' e^{-3x} + 9u e^{-3x}) + 6(u' e^{-3x} - 3u e^{-3x}) + 9u e^{-3x} = 0$

$u'' e^{-3x} \cancel{- 6u' e^{-3x}} + \cancel{9u e^{-3x}} + \cancel{6u' e^{-3x}} - 18u e^{-3x} + 9u e^{-3x} = 0$

$\frac{u'' e^{-3x}}{e^{-3x}} = \frac{0}{e^{-3x}}$

$u'' = 0$

$\int u'' = \int 0$

$u' = C_1$

$\int u'\,dx = \int c_1\,dx$

$\boxed{u = C_1 x + C_2}$

guess $y = u \cdot e^{-3x}$

$y = (C_1 x + C_2)e^{-3x}$

$y = C_1 x e^{-3x} + C_2 e^{-3x}$

two basic solutions.
one we already knew: e^{-3x}
one is new: xe^{-3x}

STEP3: if the characteristic equation has a repeated root r_1,
general solution is $y = C_1 x e^{r_1 x} + C_2 e^{r_1 x}$

Ex revisited: $y'' + 6y' + 9y = 0$

step1 guess $y = e^{rx}$

step2 characteristic equation
$r^2 + 6r + 9 = 0$
$r = -3,\ r = -3$ repeated root.

step3 general solution is $\boxed{y = C_1 x e^{-3x} + C_2 e^{-3x}}$

Complex Roots

Ex: $y'' + 4y' + 13y = 0$

Step 1: guess $y = e^{rx}$

Step 2: characteristic equation

$$r^2 + 4r + 13 = 0$$

use quadratic formula:

$$r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1}$$

$$r = \frac{-4 \pm \sqrt{16 - 52}}{2}$$

$$r = \frac{-4 \pm \sqrt{-36}}{2}$$

$$r = \frac{-4 \pm 6i}{2} = \frac{-\cancel{4}^{2}}{\cancel{2}} \pm \frac{\cancel{6}^{3}i}{\cancel{2}}$$

$$\boxed{r = -2 \pm 3i}$$

thus the general solution is $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$

$r_1 = -2 + 3i$, $r_2 = -2 - 3i$

sub. into guess:

two basic solns: $y_1 = e^{(-2+3i)x}$ $\qquad y_2 = e^{(-2-3i)x}$

general solution: $y = c_1 e^{(-2+3i)x} + c_2 e^{(-2-3i)x}$

$$\boxed{\text{Euler's Formula: } e^{bi} = \cos b + i \sin b}$$

$$y = c_1 e^{-2x + 3ix} + c_2 e^{-2x - 3ix}$$

$$\rightarrow y = c_1 e^{-2x} e^{3ix} + c_2 e^{-2x} e^{-3ix}$$

apply Euler's to e^{3ix} and e^{-3ix}

$$y = c_1 e^{-2x}\left(\cos 3x + i\sin 3x\right) + c_2 e^{-2x}\left(\cos -3x + i\sin -3x\right)$$

Trig Identities $\cos(-b) = \cos b$

$\sin(-b) = -\sin b$

$$y = c_1 e^{-2x}\left(\cos 3x + i\sin 3x\right) + c_2 e^{-2x}\left(\cos 3x - i\sin 3x\right)$$

$$y = c_1 e^{-2x}\cos 3x + c_1 e^{-2x}\cdot i\sin 3x + c_2 e^{-2x}\cos 3x + c_2 e^{-2x}(-i\sin 3x)$$

$-i\sin 3x)$

$$y = c_1 e^{-2x}\cos 3x + c_2 e^{-2x}\cos 3x + c_1 e^{-2x} i\sin 3x - c_2 e^{-2x} i\sin 3x$$

$$y = e^{-2x}\cos 3x\underbrace{(c_1 + c_2)}_{c_3} + e^{-2x}\sin 3x\underbrace{(c_1 i - c_2 i)}_{c_4}$$

general solution

$$\boxed{y = c_3 e^{-2x}\cos 3x + c_4 e^{-2x}\sin 3x}$$

$$\boxed{y = e^{-2x}\left(c_3\cos 3x + c_4\sin 3x\right)}$$

to $y'' + 4y' + 13y = 0$, char eq. roots: $r = -2 \pm 3i$

STEP 3 If the characteristic equation has complex roots $r = \lambda \pm i\cdot\omega$,

the general solution is:

$$y = e^{\lambda x}\left(c_1\cos\omega x + c_2\sin\omega x\right)$$