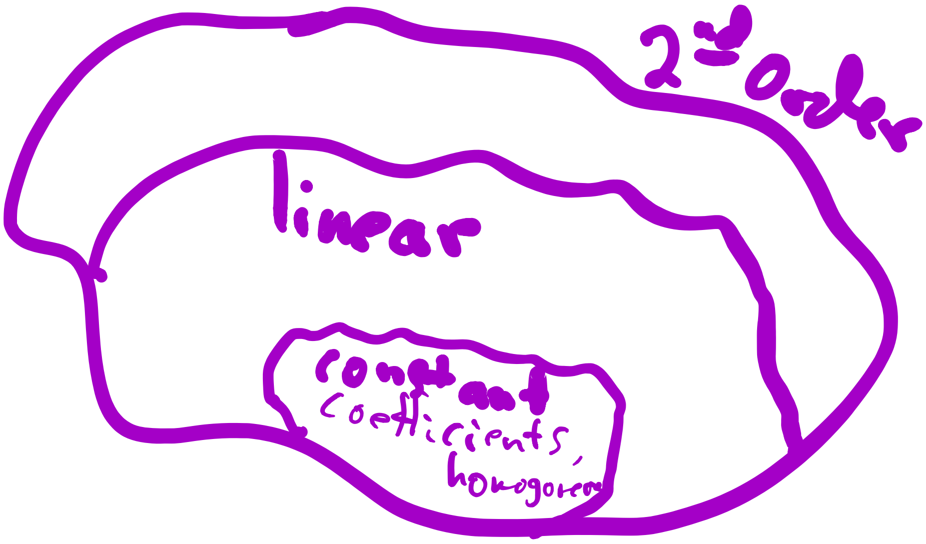

Defn a second-order differential equation is **linear** if it can be written in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

(If its not linear but its second-order, it's usually hard – we won't be studying them here)

Ex: a) $y'' - y = 0$ Is it linear? YES

$p(x) = 0$

$q(x) = -1$

$f(x) = 0$

b) verify if $y_1 = e^x$ is a solution.

YES its a solution

b) $y = e^x$

$y' = e^x$

$y'' = e^x$

Sub:

$y'' - y = 0$

$e^x - e^x = 0$

$0 = 0$ ✓

Ques: is $y = 4e^x$ a solution?

$y' = 4e^x$

$y'' = 4e^x$

sub: $y'' - y = 0$

$4e^x - 4e^x = 0$

$0 = 0$ ✓

$0 = 0$ ✓

another solution? $y = e^{-x}$?

$y' = -1 \cdot e^{-x} = -e^{-x}$

$y'' = -(-e^{-x}) = e^{-x}$

sub: $y'' - y = 0$

$e^{-x} - e^{-x} = 0$

$0 = 0$ ✓

YES

FACT: any multiple (by a constant) of e^x will be a solution:

$y = e^x$, $y = 4e^x$, $y = -e^x$, $y = \frac{1}{2}e^x$

FACT: any constant multiple of e^{-x} is a solution:

$y = e^{-x}$, $y = 5e^{-x}$, $y = -\frac{1}{2}e^{-x}$

FACT: all expressions of the form:

$$\boxed{y = c_1 e^x + c_2 e^{-x}}$$

will be solutions.

this is the most

general solution to $y'' - y = 0$

Second Order linear homogeneous equations have two basic solutions y_1, y_2

and the general solution is given by the linear combination of y_1 and y_2:

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

Defn a second-order linear homogeneous differential equation with constant coefficients has form:

$$ay'' + by' + cy = 0$$

(a, b, c are constants)

How to solve:

if we are given $ay'' + by' + cy = 0$

we guess a solution $\boxed{y = e^{rx}}$ for some constant r.

plug in: $y' = re^{rx}$

$y'' = r \cdot re^{rx} = r^2 e^{rx}$

substitute:

$ay'' + by' + cy = 0$

$a \cdot (r^2 \underline{e}^{rx}) + br\underline{e}^{rx} + c\underline{e}^{rx} = 0$

factor out e^{rx}:

$e^{rx}(ar^2 + br + c) = 0$

called the characteristic equation

so either $\cancel{e^{rx} = 0}$ or $\boxed{ar^2 + br + c = 0}$

this is a quadratic equation

two solutions, r_1 and r_2

Fact: If the characteristic equation has two real roots r_1 and r_2, then the basic solutions to the original differential equation are:

$y_1 = e^{r_1 x}$

$y_2 = e^{r_2 x}$

and the general solution is

$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

Ex: solve $\boxed{y'' - y = 0}$ using the characteristic equation.

$$ay'' + by' + cy = 0 \qquad \begin{matrix} a = 1 \\ b = 0 \\ c = -1 \end{matrix}$$

guess $\boxed{y = e^{rx}}$, substitute and solve:

$$ar^2 + br + c = 0$$

$$r^2 + 0r - 1 = 0$$

$$r^2 - 1 = 0$$

$$ {}^{+1} \qquad {}^{+1}$$

$$r^2 = 1$$

$$\sqrt{r^2} = \pm\sqrt{1} = \pm 1$$

$$\boxed{r = \pm 1} \qquad r_1 = 1,\ r_2 = -1$$

basic solutions: $y_1 = e^x$
$y_2 = e^{-x}$

general solution:

$$\boxed{y = C_1 e^x + C_2 e^{-x}}$$

Ex: find the particular solution with $y(0) = 1$, $y'(0) = 3$

substitute $y(0)=1$

$$1 = c_1 e^{0} + c_2 e^{-0}$$

$$\boxed{1 = c_1 + c_2}$$

find y', then plugin $y'(0)=3$:

$$y = c_1 e^{x} + c_2 e^{-x}$$

$$\boxed{y' = c_1 e^{x} - c_2 e^{-x}}$$

plug in: $y'(0)=3$

$$3 = c_1 e^{0} - c_2 e^{-0}$$

$$\boxed{3 = c_1 - c_2}$$

solve the system of two equations:

$$1 = C_1 + C_2$$

$$3 = C_1 - C_2$$ add

$$4 = 2C_1 + 0$$

$$\frac{4}{2} = \frac{2C_1}{2}$$

$$\boxed{2 = C_1}$$

substitute into $1 = C_1 + C_2$

$$1 = 2 + C_2$$

$$-2 \quad -2$$

$$\boxed{-1 = C_2}$$

Particular solution:

$$\boxed{y = 2e^{x} - e^{-x}}$$

Ex: solve

$$y'' + 6y' + 5y = 0,$$

$$y(0) = 3, \quad y'(0) = -1$$

$$y'' + 6y' + 5y = 0$$

guess $y = e^{rx}$, substitute.

characteristic equation:

$$r^2 + 6r + 5 = 0$$

Solve.

factor:

$$(r+5)(r+1) = 0$$

$r+5=0$ $\qquad$ $r+1=0$

$\boxed{r=-5}$ $\qquad$ $\boxed{r=-1}$

two real roots

general solution:

$$y = c_1 e^{-5x} + c_2 e^{-x}$$

sub $y(0) = 3$

$$3 = c_1 e^{-5 \cdot 0} + c_2 e^{-0}$$

$$\boxed{3 = C_1 + C_2}$$

$$y' = -5c_1 e^{-5x} - c_2 e^{-x}$$

sub $y'(0) = -1$

$$-1 = -5c_1 e^{-5 \cdot 0} - c_2 e^{-0}$$

$$\boxed{-1 = -5c_1 - C_2}$$

$$3 = c_1 + c_2$$
$$-1 = -5c_1 - c_2$$

add.

$$2 = -4c_1 + 0$$

$$2 = -4c_1$$

$$c_1 = \frac{2}{-4} = \boxed{-\frac{1}{2}}$$

$$3 = -\frac{1}{2} + c_2$$
$$+\frac{1}{2} \qquad +\frac{1}{2}$$

$$\frac{7}{}$$

$$\boxed{\frac{\quad}{2} = C_2}$$

$$\boxed{y = -\frac{1}{2}e^{-5x} + \frac{7}{2}e^{-x}}$$

ANS

Office Hours

Suppose that we use the Improved Euler's method to approximate the solution to the differential equation

$$\frac{dy}{dx} = x - 1.5y; \qquad y(0.2) = 9.$$

Let $f(x, y) = x - 1.5y$.
We let $x_0 = 0.2$ and $y_0 = 9$ and pick a step size $h = 0.25$. The improved Euler method is the the following algorithm. From (x_n, y_n), our approximation to the solution of the differential equation at the n-th stage, we find the next stage by computing the x-step $x_{n+1} = x_n + h$, and then k_1, the slope at (x_n, y_n). The predicted new value of the solution is $z_{n+1} = y_n + h \cdot k_1$. Then we find the slope at the predicted new point $k_2 = f(x_{n+1}, z_{n+1})$ and get the corrected point by averaging slopes

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2).$$

Complete the following table:

n	x_n	y_n	k_1	z_{n+1}	k_2
0	0.2	9	−13.3	5.675	−8.0625
1					
2					
3					
4					

The exact solution can also be found for the linear equation. Write the answer as a function of x.
$y(x) =$

Thus the actual value of the function at the point $x = 1.2$ is
$y(1.2) =$.

part II: solve

$y' = x - 1.5y$

first order linear nonhomogeneous.

Complete the following table:

n	x_n	y_n	k_1	z_{n+1}	k_2
0	0.2	9	−13.3	5.675	−8.0625
1	0.45	6.329875			
2	0.7				
3	0.95				
4	1.2				

$y' = f(x,y) = x - 1.5y$, $h = 0.25$

(x_i, y_i)
$(0.2, 9)$ initial point

$K_1 = f(0.2, 9) = 0.2 - 1.5(9) = -13.3$

$K_1 = f(x_i, y_i)$

$Z = 9 + (-13.3)(0.25) = 5.675$

$\underline{Z = y_i + K_1 h}$

$K_2 = f(.45, 5.675) = .45 - 1.5(5.675)$
$= -8.0625$

$K_2 = (x_{i+1}, Z)$

$y_{n+1} = 9 + \frac{-13.3 + -8.0625}{2} \cdot (.25)$

$y_{n+1} = y_i + \frac{K_1 + K_2}{2} \cdot h$

$y_{n+1} = 6.3296875$

$(.45, 6.3296875)$ initial point

$x_{i+1} = .45 + .25 = .7$

$K_1 = f(.45, 6.3296875) = .45 - 1.5(6.3296875)$
$= -9.04453125$

$Z = 6.3296875 + (-9.04453125)(.25)$

$= 4.0685546875$

$k_2 = f(.7, 4.0685546875) =$

$.7 - 1.5(4.0685546875) =$

$= -5.402832031 25$

$y_{n+1} = 6.3296875 + \frac{(-9.0445312 5 - 5.402832)}{2} \cdot .25$

$y_{n+1} = 4.523767 08984$

$(.7, 4.52376708984)$ initial point

Part II - solve.

$y' = x - 1.5y \qquad y(0.7) = 9$

$y' + 1.5y = x$

STEP 1: solve complementary: (single solution)

$y' + 1.5y = 0$

$\quad -1.5y \quad -1.5y$

$\frac{y'}{y} = \frac{-1.5y}{y}$

$\int \frac{y'}{y} = \int -1.5$

$\ln|y| = -1.5x + C \qquad$ choose $c = 0.$

$\ln|y| = -1.5x + 0$

$e^{\ln|y|} = e^{-1.5x}$

$|y| = e^{-1.5x}$

$y = \pm e^{-1.5x}$ choose $+$

$\boxed{y_1 = e^{-1.5x}}$

STEP 2

guess $y = u \cdot y_1$

$\boxed{y = u \cdot e^{-1.5x}}$

find y':

$y' = u' \cdot e^{-1.5x} + u \cdot (-1.5) e^{-1.5x}$ product rule.

substitute into $y' + 1.5y = x$

$u' \cdot e^{-1.5x} \cancel{- 1.5 u e^{-1.5x}} + \cancel{1.5 u e^{-1.5x}} = x$

$u' e^{-1.5x} = x \quad (e^{1.5x})$

$\int u' dx = \int x e^{1.5x} dx$

$\int x e^{1.5x} dx$ by parts

$u = x \qquad dv = e^{1.5x} dx$

$du = dx \qquad v = \frac{1}{1.5} e^{1.5x}$

$\int x e^{1.5x} dx = x \cdot \frac{1}{1.5} e^{1.5x} - \int \frac{1}{1.5} e^{1.5x} dx$

$= \frac{1}{1.5} x e^{1.5x} - \frac{1}{1.5}\left(\frac{1}{1.5} e^{1.5x}\right)$

$\frac{1}{1.5} e^{1.5x}$

check - take deriv.

$$\frac{d}{dx}e^{1.5x} = 1.5e^{1.5x}$$

$$= \frac{2}{3}xe^{1.5x} - \frac{2}{3}\left(\frac{2}{3}e^{1.5x}\right)$$

$$= \frac{2}{3}xe^{1.5x} - \frac{4}{9}e^{1.5x}$$

$$u = \frac{2}{3}xe^{1.5x} - \frac{4}{9}e^{1.5x} + C$$

substitute into $y = u \cdot e^{-1.5x}$

$$y = \left(\frac{2}{3}xe^{1.5x} - \frac{4}{9}e^{1.5x} + c\right)e^{-1.5x}$$

$$\boxed{y = \frac{2}{3}x - \frac{4}{9} + ce^{-1.5x}}$$

general solution.

to find c and get a particular solution plug in $y(0.2) = 9$

$$9 = \frac{2}{3}(0.2) - \frac{4}{9} + ce^{-1.5(0.2)}$$

$$9 = \underbrace{0.133333 - .444444}_{} + c(.740818)$$

$$9 - 0.133333 + .44444 = c(.740818)$$

$$\boxed{c = 12.568689/}$$

$$\boxed{y = \frac{2}{3}x - \frac{4}{9} + 12.568689/e^{-1.5x}}$$

particular solution.