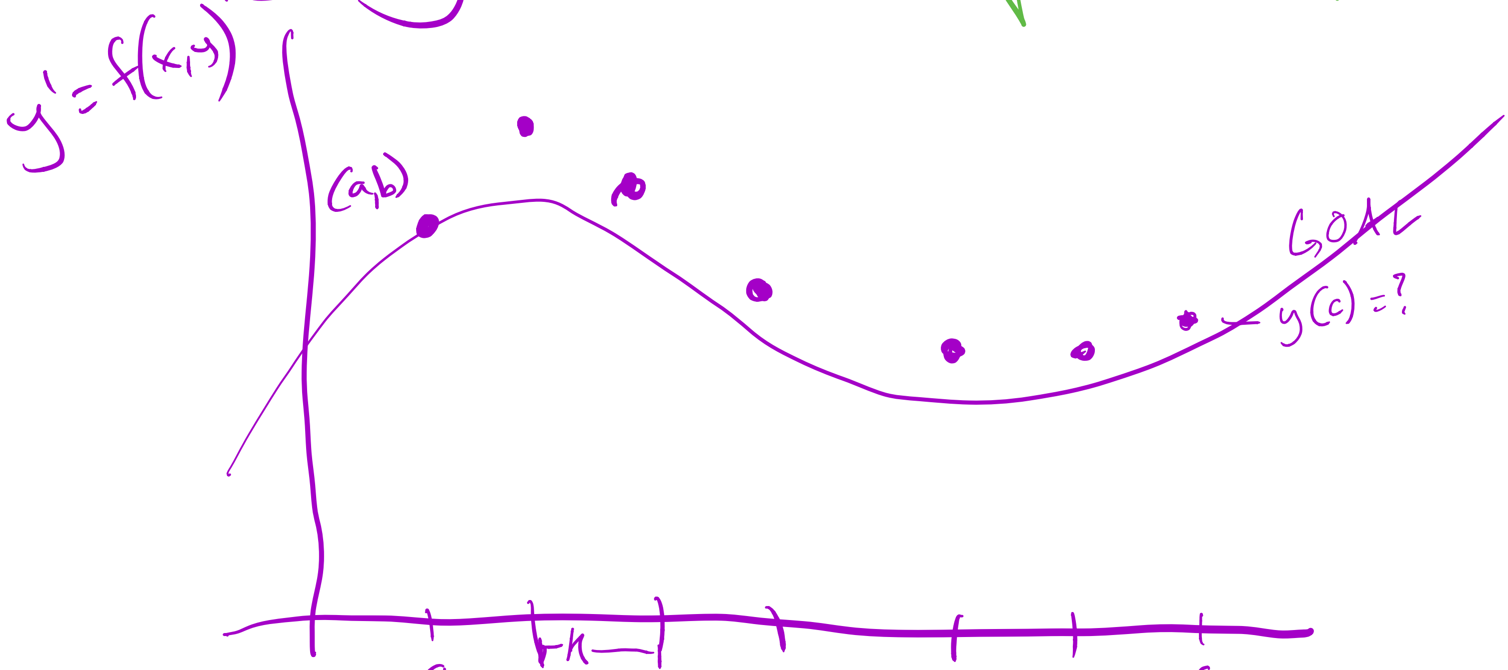
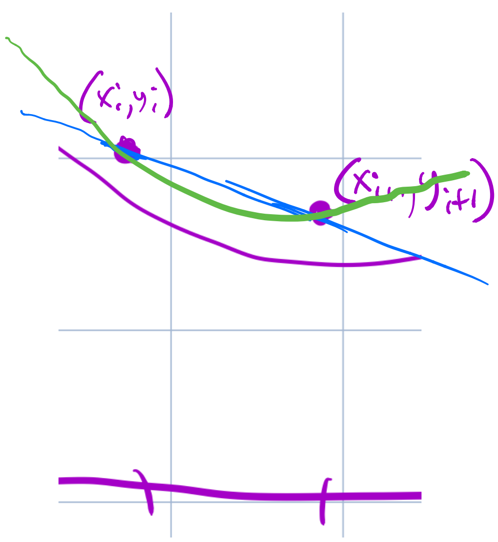
Big Idea — use a parabola.



$n = \#$ of steps

$$h = \frac{c-a}{n}$$



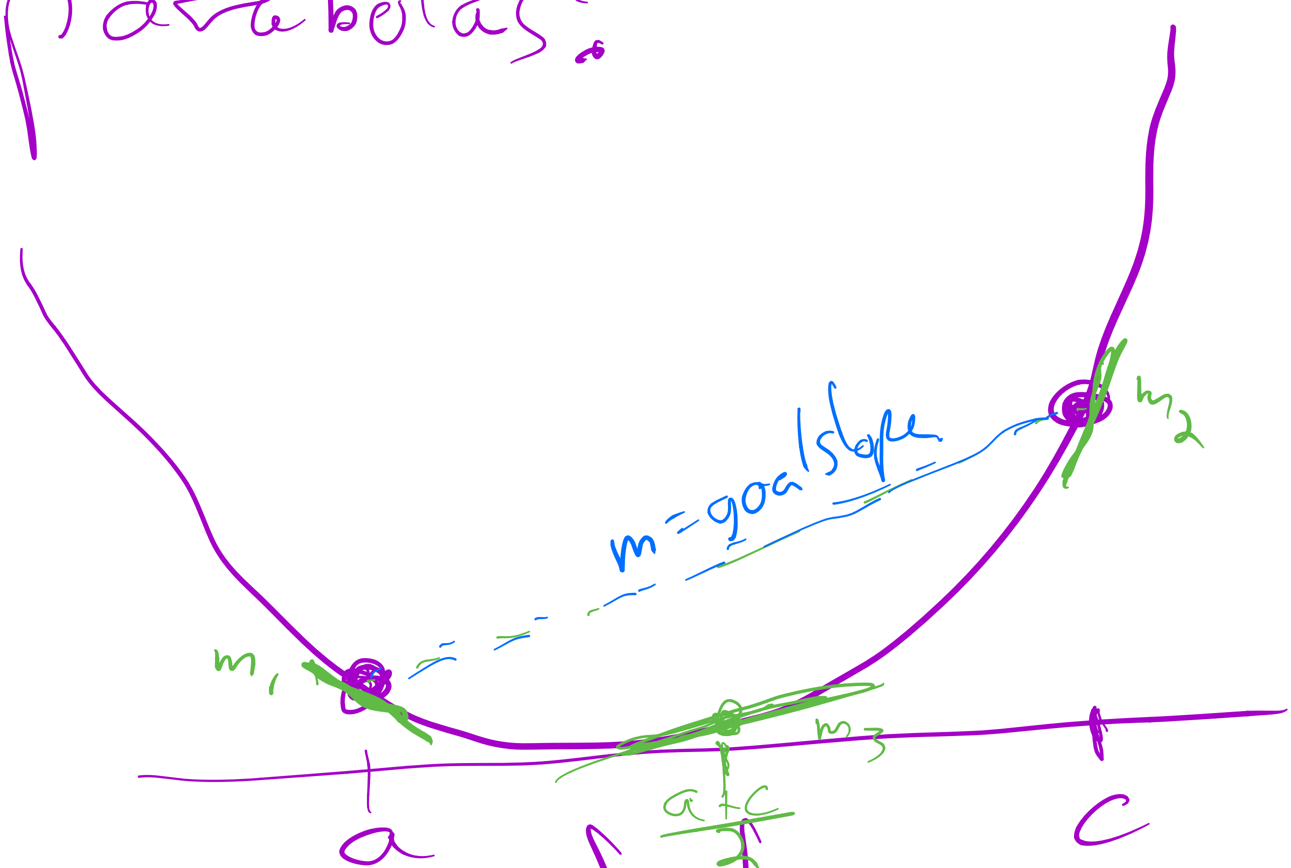
Euler's / Improved Euler's

→ approximate the curve with a line.

Runge-Kutta → approximate the curve with a parabola instead.

GIVES A MORE ACCURATE ESTIMATE for the next point.

Cool Fact about parabolas:



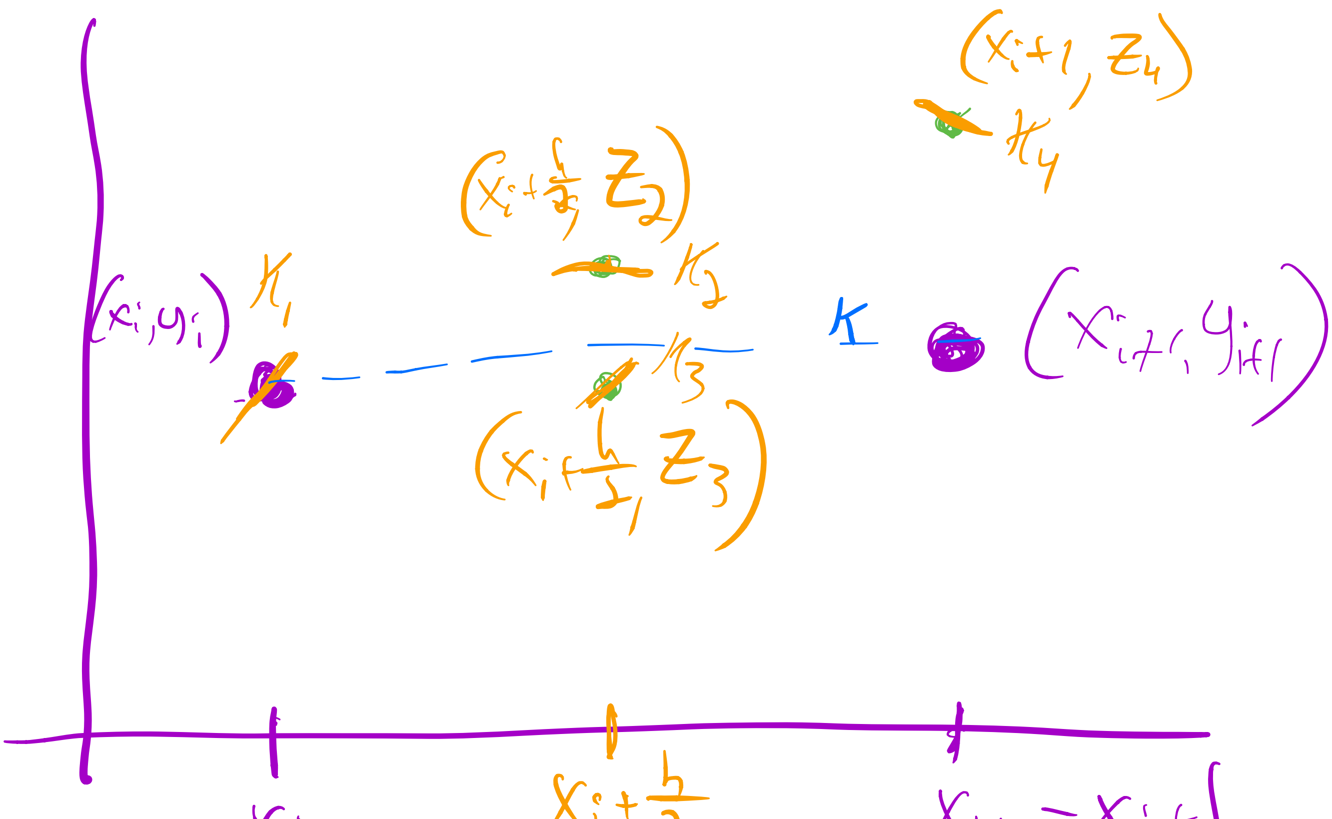
on a parabola

Cool Fact:

$$m = \frac{m_1 + 4m_3 + m_2}{6}$$

gives the slope from $(a, f(a))$ to $(c, f(c))$

How we use this in Runge-Kutta:

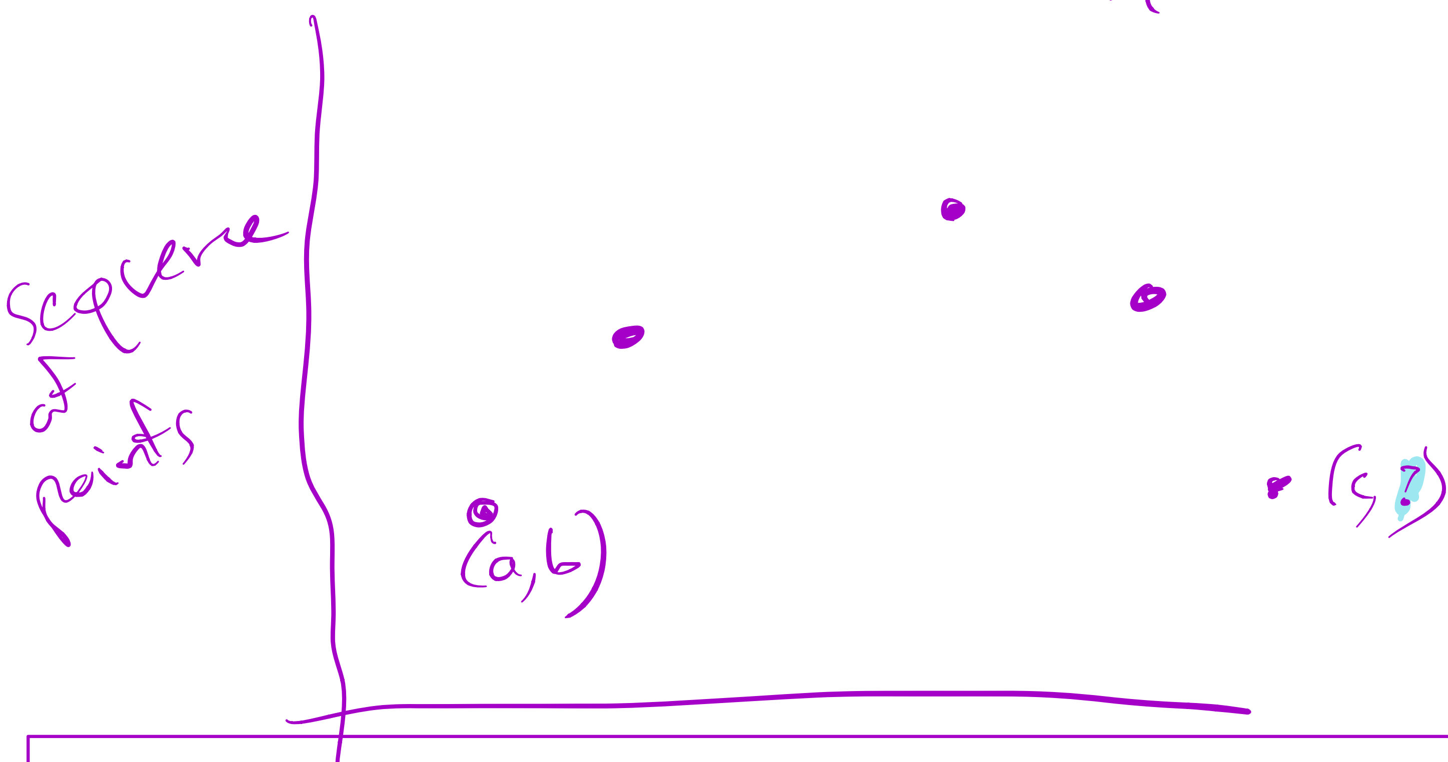


x_i $\quad x_i + \frac{h}{2}$ $\quad x_{i+1} = x_i + h$

$$K = \frac{K_1 + 2K_2 + 2K_3 + K_4}{6}$$

Runge-Kutta Method

Given $y' = f(x,y)$, $y(a) = b$, to estimate $y(c)$ using n steps and step size $h = \frac{c-a}{n}$



How do we go from one

point (x_i, y_i) to the next?

Start with (x_i, y_i)

$$x_{i+1} = x_i + h$$

$$K_1 = f(x_i, y_i)$$

$$z_2 = y_i + K_1 \cdot \frac{h}{2}$$

second point:

$$\left(x_i + \frac{h}{2}, z_2\right)$$

$$K_2 = f\left(x_i + \frac{h}{2}, z_2\right)$$

third point:

$$\left(x_i + \frac{h}{2}, z_3\right)$$

$$z_3 = y_i + K_2 \cdot \frac{h}{2}$$

$$K_3 = f\left(x_i + \frac{h}{2}, z_3\right)$$

fourth point

$$(x_{i+1}, z_4)$$

$$z_4 = y_i + K_3 \cdot h$$

$$K_4 = f(x_{i+1}, z_4)$$

$$\boxed{y_{i+1} = y_i + \frac{K_1 + 2K_2 + 2K_3 + K_4}{6} \cdot h}$$

new point

$$\boxed{(x_{i+1}, y_{i+1})}$$

Example 1: Consider the initial value problem y'+2y=x^3 e^(-2x), y(0)=1. Approximate the value of y(0.6) using a step size of 0.3.

$h = 0.3$

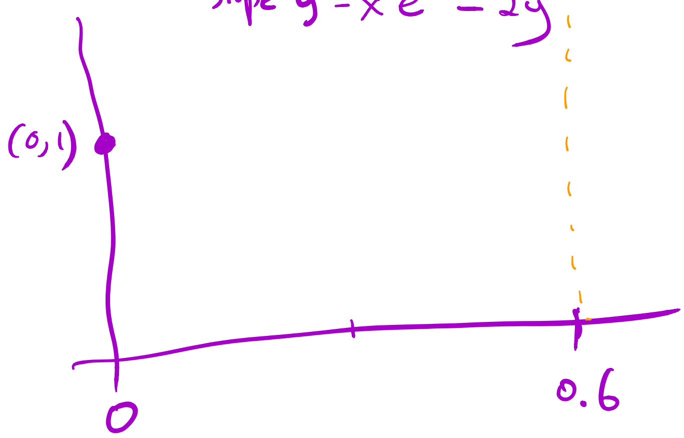
$n = \frac{c-a}{h}$

$n = \frac{.6-0}{.3} = 2 \rightarrow n = 2$

initial point $\overset{(a,b)}{(0,1)}$

goal point $(0.6, ?)$

slope $y' = x^3 e^{-2x} - 2y$ $f(x,y)$



$y' + 2y = x^3 e^{-2x}$

$\quad -2y \qquad -2y$

$$\boxed{y' = f(x,y) = x^3 e^{-2x} - 2y}$$

i	h	x	y	K_1	Z_2	K_2	Z_3	K_3	Z_4	K_4	y_{i+1}
0	.3	0	1	-2	0.7	-1.3974997	.790375				
1	.3	.3									
2	.3	.6									

start with (0,1)

$$K_1 = f(0,1) = 0^3 e^{-2\cdot 0} - 2\cdot 1 = -2$$

$$Z_2 = y_i + K_1\cdot\frac{h}{2} = 1 + (-2)\cdot\frac{.3}{2} = 0.7$$

$\left(x_i + \frac{h}{2}, 0.7\right)$

$\left(0 + \frac{.3}{2}, 0.7\right)$

second point $\rightarrow$ $(.15, 0.7)$

$$K_2 = f(.15, 0.7) = (.15)^3 e^{-2(.15)} - 2(0.7)$$

$$= -1.3974997$$

$$Z_3 = y_i + K_2\cdot\frac{h}{2} = 1 + (-1.3974997)\cdot\frac{.3}{2}$$

$$= .790375$$