**Some practice problems:**

For each problem below, clearly identify the type of equation: Linear Equation, Separable Equation, Bernoulli Equation, Homogeneous Equation ("y/x"), or Exact Equation. Find the general solution (give an explicit solution unless instructed otherwise). If there is an initial condition given, also find the particular solution satisfying the condition.

1. $\frac{1}{t^3}\frac{dy}{dt} = \frac{1}{y^4}$
2. $ty' - 5y = t^4y^3$
3. $y' + 6y = -\left(2t + 12e^{-5t}\right),\ y(0) = -10$
4. (1) * $15x^2y^4 + e^x \sin y + \left(20x^3y^3 + e^x \cos y + \frac{1}{y+1}\right)y' = 0$, solve implicitly
5. $y' = (-6 - 3x)y^2$, $y(0) = \frac{2}{9}$ (give the interval of validity)
6. $(t^2 + 5)\frac{dy}{dt} = t(81 + y^2)$
7. (2) * $\frac{dy}{dx} = \frac{x^2+5y^2}{7xy}$, solve implicitly → don't have to solve for y.
8. * $xy' = 8x\sin(3x) - y$

# Exam 1 Review ANSWER KEY

*If you discover an error please let me know, either in class, on the OpenLab, or by email to jreitz@citytech.cuny.edu. Corrections will be posted on the "Exam Reviews" page.*

1. Separable. $y = \left(\frac{5}{4}t^4 + c\right)^{1/5}$
2. Bernoulli Equation. $y = \pm\sqrt{\frac{7t^{10}}{c - t^{14}}}$
3. Linear. General solution: $y = ce^{-6t} - \frac{t}{3} - 12e^{-5t} + \frac{1}{18}$.

   Particular solution when $y(0) = -10$: $y = \frac{35}{18}e^{-6t} - \frac{t}{3} - 12e^{-5t} + \frac{1}{18}$
4. Exact. $5x^3y^4 + e^x \sin y + \ln|y + 1| = c$
5. Separable. General solution: $y = \frac{2}{3x^2+12x+c}$

   Particular solution when $y(0) = \frac{2}{9}$: $y = \frac{2}{3x^2+12x+9}$, interval of validity: $(-1, \infty)$
6. Separable. $y = 9\tan\left(\frac{9}{2}ln|t^2 + 5| + C\right)$
7. Homogeneous. $-\frac{7}{4}\ln|1 - 2\frac{y^2}{x^2}| = \ln|x| + C$
8. Linear. $y = \frac{c}{x} + \frac{8\sin(3x)}{9x} - \frac{8}{3}\cos(3x)$

$y' + p(x)y = f(x)$ linear

4. $\underbrace{15x^2y^4 + e^x \sin y}_{M} + \underbrace{\left(20x^3y^3 + e^x \cos y + \frac{1}{y+1}\right)}_{N} y' = 0$, solve implicitly

$\frac{d}{dx}(5x+7) = 5 + 0$

Exact.

Step 1: check $M_y = N_x$

$M_y = \frac{\partial}{\partial y}(15x^2y^4 + e^x \sin y) = 15x^2 \cdot 4y^3 + e^x \cos y = 60x^2y^3 + e^x \cos y$

$N_x = \frac{\partial}{\partial x}\left(20x^3y^3 + e^x \cos y + \frac{1}{y+1}\right) = 20y^3 \cdot 3x^2 + e^x \cos y + 0 = 60x^2y^3 + e^x \cos y$

It's exact since $M_y = N_x$.

Step 2: $\int M\,dx = \int 15x^2y^4 + e^x \sin y \, dx$

$F(x,y) = \cancel{15}^{5} y^4 \frac{x^3}{\cancel{3}} + e^x \sin y + \phi(y)$

Step 3: $\frac{\partial}{\partial y} F = N$

$\frac{\partial}{\partial y}\left(5y^4x^3 + e^x \sin y + \phi(y)\right)$

$\underbrace{5x^3 \cdot 4y^3 + e^x \cos y + \phi'(y)}_{F_y \text{ or } \frac{\partial}{\partial y}F}$

$20x^3y^3 + e^x \cos y + \phi'(y) = 20x^3y^3 + e^x \cos y + \frac{1}{y+1}$

Solve for $\phi'(y)$

$\int \phi'(y)\,dy = \int \frac{1}{y+1}\,dy$

$\phi(y) = \ln|y+1| + C$

$\boxed{F(x,y) = 5y^4x^3 + e^x \sin y + \ln|y+1|}$

Set $F(x,y) = c$ to get implicit solution.

$\boxed{5y^4x^3 + e^x \sin y + \ln|y+1| = C}$ Implicit Solution.

7. $\frac{dy}{dx} = \frac{x^2+5y^2}{7xy}$, solve implicitly homogeneous $\frac{y}{x}$?

$y' = \frac{x^2}{7xy} + \frac{5y^2}{7xy}$

$y' = \frac{x}{7y} + \frac{5y}{7x}$

$\boxed{y' = \frac{1}{7}\left(\frac{x}{y}\right) + \frac{5}{7}\left(\frac{y}{x}\right)}$

let $y_1 = x$

guess $y = u \cdot x \longrightarrow u = \frac{y}{x}$

$y' = u \cdot 1 + u'x = u + u'x$

substitute into

$+5u$
$-u$

$u + u'x = \frac{1}{7} \cdot u^{-1} + \frac{5}{7} \cdot u$ ← should be separable. Separate!

$-u \qquad -u$

$u'x = \frac{1}{7u} + \left(\frac{5}{7} - 1\right)u$

$u'x = \frac{1}{7u} + \frac{-2u}{7} \cdot \frac{u}{u}$

$\frac{1}{x} \cdot \frac{7u}{1-2u^2} \cdot u'x = \frac{1-2u^2}{7u} \cdot \frac{7u}{1-2u^2} \cdot \frac{1}{x}$

$\int \frac{7u}{1-2u^2}\, u'\,dx = \int \frac{1}{x}\,dx$

$$= \ln|x| + c$$

$$\int \frac{7u}{1-2u^2}\, du$$

w-substitution

$w = 1-2u^2$

$dw = -4u\, du$

$\frac{dw}{-4} = u\, du$

$$\int \frac{7}{-4}\frac{dw}{(w)} = -\frac{7}{4}\int \frac{1}{w}\, dw$$

$$= -\frac{7}{4}\ln|w|$$

$$= -\frac{7}{4}\ln|1-2u^2|$$

$$\boxed{-\frac{7}{4}\ln|1-2u^2| = \ln|x| + C}$$

Solve for u, then plug in to $y = ux$

$u = \frac{y}{x}$

sub $u = \frac{y}{x}$: $\boxed{-\frac{7}{4}\ln\left|1-2\frac{y^2}{x^2}\right| = \ln|x| + C}$ Final answer.

$$\ln|1-2u^2| = -\frac{4}{7}\left(\ln|x| + C\right)$$

$$\ln|1-2u^2| = -\frac{4}{7}\ln|x| + C_1$$

$$e^{\ln|1-2u^2|} = e^{-\frac{4}{7}\ln|x| + C_1}$$

$$|1-2u^2| = e^{-\frac{4}{7}\ln|x|}\, e^{C_1}$$

let $C_2 = e^{C_1}$

$$= e^{\ln|x^{-\frac{4}{7}}|} \cdot C_2$$

$$|1-2u^2| = |x^{-\frac{4}{7}}| \cdot C_2$$

$$1-2u^2 = \pm x^{-\frac{4}{7}} \cdot C_2$$

$C_3 = \pm C_2$

$$1-2u^2 = C_3 x^{-\frac{4}{7}}$$

$-1$ $-1$

$$\frac{-2u^2}{-2} = \frac{C_3 x^{-\frac{4}{7}} - 1}{-2}$$

$C_4 = \frac{C_3}{-2}$

$$u^2 = \frac{C_3}{-2} x^{-\frac{4}{7}} + \frac{1}{2}$$

$$u = \pm\sqrt{C_4 x^{-\frac{4}{7}} + \frac{1}{2}}$$

plug into our guess $y = ux$

$$\boxed{y = \left(\pm\sqrt{C_4 x^{-\frac{4}{7}} + \frac{1}{2}}\right) \cdot x}$$

# Recall:

Example 1: A population of field mice inhabit a rural area. Some owls live in the area. Let's track the population p(t) over time (t in months).

- every month, each pair of mice produces one new mouse.
- every night, the owls eat 15 mice. 450 mice per month.

How many mice are killed each month?
How many mice are born each month?
Does the population go up, or down?

each month, the change in population is

$\boxed{\frac{dp}{dt} = \frac{1}{2}p - 450}$ ← a differential equation describing mouse population.

## General Setup - Modeling Population

$$\frac{dp}{dt} = rp - K$$

$t$ = time

$p(t)$ = population at time $t$.

$r$ = growth rate, reproduction rate

$K$ = predation rate (# killed by predators).

**Observe:** if I give you r, K,
can you find general formula for $p(t)$? plug in r, K, solve the linear differential equation.

if I give you the initial population:
ex: $p(0) = 600$
you can find value of c and get a particular solution $p(t)$

Equilibrium solution – when reproduction + predation are balanced so $p(t)$ remains constant.

To <u>find</u> equilibrium population, set $\frac{dp}{dt} = 0$.

# Office Hours

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5. $y' = (-6 - 3x)y^2,\ y(0) = \frac{2}{9}$ (give the interval of validity)
6. $(t^2 + 5)\frac{dy}{dt} = t(81 + y^2)$
7. $\frac{dy}{dx} = \frac{x^2+5y^2}{7xy}$, solve implicitly → don't have to solve for y
8. $xy' = 8x\sin(3x) - y$

linear: $y' + p(x)y = f(x)$

bernoulli: $y' + p(x)y = f(x)\cdot y^n$ $(n \neq 0, 1)$

2. $\frac{ty' - 5y}{t} = \frac{t^4y^3}{t}$

$$\boxed{y' - \frac{5}{t}y = t^3y^3}$$

this is Bernoulli

**Step 1** complementary:

$$y' - \frac{5}{t}y = 0$$

separate variables

$$+\frac{5}{t}y \qquad +\frac{5}{t}y$$

$$\frac{y'}{y} = \frac{\frac{5}{t}y}{y}$$

$$\int \frac{1}{y}y'\,dt = \int \frac{5}{t}\,dt$$

$$\ln|y| = 5\ln|t| + c$$

just need a single solution, choose $c=0$

$$\ln|y| = 5\ln|t|$$

$$e^{\ln|y|} = e^{5\ln|t|}$$

$$|y| = e^{\ln|t^5|}$$

$$|y| = |t^5|$$

$$y = \pm t^5$$

choose +

$$\boxed{y_1 = t^5}$$

**Steps**

1. Find $y_1$ a solution to complementary eq.
2. guess $y = u\cdot y_1$, substitute, simplify, integrate to find $u$
3. plug in $y = u\cdot y_1$

substitute $y_1$, $y'$ back into the original equation

$5\ln|t|$

$\ln(|t|^5) = \ln|t^5|$

$\int \frac{1}{t}\,dt = \ln|t|$

$\int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = t^{-1}$

Guess: $y = u\cdot y_1$

$$\boxed{y = u \cdot t^5}$$

Substitute into $\boxed{y' - \frac{5}{t} y = t^3 y^3}$ original equation

Find $y' = u \cdot 5t^4 + u' \cdot t^5$

$$5ut^4 + u't^5 - \frac{5}{t}(u \cdot t^5) = t^3 (u \cdot t^5)^3$$

$$5ut^4 + u't^5 - 5ut^4 = t^3 u^3 t^{15}$$

Goal: integrate both sides (need to separate)

$$\frac{u't^5}{t^5} = \frac{t^{18} u^3}{t^5}$$

$$\frac{u'}{u^3} = \frac{t^{13} u^3}{u^3}$$

$$\int \frac{1}{u^3} u' dt = \int t^{13} dt$$

$$\int \frac{1}{x^3} dx =$$

$$\int x^{-3} dx = \frac{x^{-2}}{-2}$$

$$\int u^{-3} u' dt = \frac{t^{14}}{14} + C$$

$$\frac{u^{-2}}{2} = \frac{t^{14}}{14} + C$$

solve for u:

$$u^{-2} = \frac{2t^{14}}{\cancel{14}\, 7} + 2C$$

$C_1 = 2C$

$$u^{-2} = \frac{t^{14}}{7} + C_1$$

$$\frac{1}{u^2} = \frac{t^{14}}{7} + C_1 \cdot \frac{7}{7}$$

$$\frac{1}{u^2} = \frac{t^{14}}{7} + \frac{7C_1}{7}$$

$$\frac{1}{u^2} = \frac{t^{14} + 7c_1 c_2}{7} \qquad c_2 = 7c_1$$

$$\frac{1}{u^2} = \frac{t^{14} + c_2}{7}$$

flip both sides:

$$\frac{u^2}{1} = \frac{7}{t^{14} + C_2}$$

$$u^2 = \frac{7}{t^{14} + C_2}$$

$$\boxed{u = \pm\sqrt{\frac{7}{t^{14} + C_2}}}$$

plug in $y = u \cdot y_1$

* Final answer

$$\boxed{y = \left(\pm\sqrt{\frac{7}{t^{14} + C_2}}\right) \cdot t^5}$$

$y = \pm\sqrt{\frac{7t^{10}}{c - t^{14}}}$ ← Answer Key

8. $\frac{xy'}{x} = \frac{8x\sin(3x) - y}{x}$ $\qquad y' + p(x)y = f(x)$

$$y' = 8\sin(3x) - \frac{y}{x}$$

$$+\frac{y}{x} \qquad +\frac{y}{x}$$

$$y' + \frac{y}{x} = 8\sin(3x)$$

$$\boxed{y' + \frac{1}{x}y = 8\sin(3x)}$$

linear, nonhomogeneous $(\neq 0)$

Step1: find $y_1$ a solution to complementary equation.

Step2: guess $y = u \cdot y_1$ substitute $y, y'$ into original, simplify, integrate to find $u$.

Step3: substitute

$y = u \cdot y_1$ // solution