Exact Equations

Today: two types of differentiation

Implicit Differentiation

y is a function of x

Partial Differentiation

treat y as constant when differentiating with respect to x.

Example 1: Given the differential equation $6x^5 + 2xy^2 + (2yx^2 + 5y^4)\frac{dy}{dx} = 0$,

verify that this equation $\underline{x^6 + x^2y^2 + y^5 = C}$ is an implicit solution.

To check, take derivative of the equation (Implicit Diff.)

$$\frac{d}{dx}\left(x^6 + x^2y^2 + y^5\right) = \frac{d}{dx} C$$

$$6x^5 + 2xy^2 + \underline{x^2 \cdot 2y \cdot y' + 5y^4 \cdot y'} = 0$$

$$6x^5 + 2xy^2 + (x^2 \cdot 2y + 5y^4) \cdot y' = 0$$

$$6x^5 + 2xy^2 + (2yx^2 + 5y^4) \cdot \frac{dy}{dx} = 0 \checkmark$$

Yes, it is a solution.

Fancy "d"

Partial Differentiation written $\frac{\partial}{\partial x}$

Treat all other variables as constant.

Ex Find a) $\frac{\partial}{\partial x}$ and b) $\frac{\partial}{\partial y}$ of $\overbrace{x^6 + x^2y^2 + y^5}^{F(x,y)}$

a) $\frac{\partial}{\partial x}\left(x^6 + x^2y^2 + y^5\right) = 6x^5 + 2xy^2 + 0 = \boxed{6x^5 + 2xy^2}$

$$\frac{d}{dx} 5x^2 = 10$$

b) $\frac{\partial}{\partial y}\left(x^6 + x^2y^2 + y^5\right) = 0 + 2x^2y + 5y^4$

$= \boxed{2x^2y + 5y^4}$

Original Diff Eq: $\overbrace{6x^5 + 2xy^2}^{M(x,y)} + \overbrace{(2yx^2 + 5y^4)}^{N(x,y)}\frac{dy}{dx} = 0$,

Defn If a differential equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact, then there is a function $F(x,y)$ with

a) $\frac{\partial}{\partial x} F = F_x = M(x,y)$

b) $\frac{\partial}{\partial y} F = F_y = N(x,y)$

c) $F(x,y) = C$ is the (implicit) general solution to the equation.

Goal: Given $M(x,y) + N(x,y)\frac{dy}{dx} = 0$, find $F(x,y)$.

How do we tell if it's exact?

$$\begin{tikzcd}
 & F(x,y) \arrow[dl, "F_x"'] \arrow[dr, "F_y"] & \\
F_x = M(x,y) \arrow[dr, "F_{xy}"'] & & N(x,y) = F_y \arrow[dl, "F_{yx}"] \\
 & M_y = N_x &
\end{tikzcd}$$

are these equal? if so, the equation is exact.

SOLVING EXACT EQUATIONS:
Given a differential equation of the form: $M(x, y) + N(x, y)y' = 0$
1. Verify that the equation is **exact** by checking that $M_y = N_x$.
2. Integrate M with respect to x to obtain $F(x, y)$. *Treat y as a constant. Don't forget to add a "constant" term* $\phi(y)$.
3. Take the partial derivative F_y and set it equal to N, solve for $\phi'(y)$.
4. Integrate $\phi'(y)$ to find $\phi(y)$
5. The general solution to the differential equation is given implicitly by: $F(x, y) = c$.

Example 3. Solve $(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$
ANS: $y\sin x + x^2e^y - y = c$

Example 4. Solve $(3xy + y^2) + (x^2 + xy)y' = 0$
NOTE: $M_y \neq N_x$ *and so this equation is not exact - this method will not work.*

$M(x,y)$ $N(x,y)$

Example 3. Solve $(y\cos x + 2xe^{y}) + (\sin x + x^2e^{y} - 1)y' = 0$

$M(x,y) = y\cos x + 2xe^y$

$N(x,y) = \sin x + x^2e^y - 1$

STEP 1 Does $M_y = N_x$?

$$\cancel{\frac{\partial}{\partial x}M = \frac{\partial}{\partial x}(y\cos x + 2xe^y) = -y\sin x + 2e^y = M_x}$$

$$\frac{\partial}{\partial y}M = \frac{\partial}{\partial y}(y\cos x + 2xe^y) = 1\cdot\cos x + 2xe^y = M_y$$

$$\frac{\partial}{\partial x}N = \frac{\partial}{\partial x}(\sin x + x^2e^y - 1) = \cos x + 2xe^y + 0 = N_x$$

check ✓ $M_y = N_x$ so the equation is exact.

STEP 2 Integrate M with respect to x:

$$\int y\cos x + 2xe^y\,dx$$

treat y as constant:

$$F(x,y) = y\sin x + x^2e^y + \phi(y)$$

some function of y (unknown)

STEP 3 Find $\frac{\partial}{\partial y}F(x,y)$, set equal to $N(x,y)$

$$\frac{\partial}{\partial y}\left(y\sin x + x^2e^y + \phi(y)\right) = \sin x + x^2e^y + \phi'(y) = F_y$$

set $F_y = N$

$$\sin x + x^2e^y + \phi'(y) = \sin x + x^2e^y - 1$$

$$-\sin x \quad -x^2e^y \qquad\qquad -\sin x \quad -x^2e^y$$

STEP 4:
$$\int \phi'(y)\,dy = \int -1\,dy$$

$$\phi(y) = -1\cdot y = -y$$

STEP 5: $F(x,y) = y\sin x + x^2e^y - y$

solution to our diffy Q is:

$$F(x,y) = C$$

$$\boxed{y\sin x + x^2e^y - y = C}$$