DAY 4: Section 2.4
Transformation of Nonlinear Equations into Separable Equations (p.62–68)
p.68: 1–4, 7–11 odd, 15–18, 23–27 odd
- Bernoulli Equations

TODAY'S TRICK: Using substitutions to turn nonlinear, nonseparable equations into separable equations.

HEADS UP: In today's lecture let's use "t" as our independent variable instead of "x"...

Defn: A **Bernoulli Equation** has the form $y' + p(t)y = f(t)y^r$, where r is any real number except 0 or 1.
STEP 1: Let $y_1(t)$ be a solution to the complementary equation $y' + p(t)y = 0$.
STEP 2: Guess a solution of the form $y = uy_1$, where $u(t)$ is some (unknown) function of t.
STEP 3: Substitute into the original equation and separate variables. Then integrate to find u.

Example 1: Solve the initial value problem $y' - y = ty^2$, y(1)=e

ANS (General): $y = \frac{e^t}{e^t(1-t)+c}$ or $y = -\frac{1}{t-1+ce^{-t}}$

ANS (IVP): $y = \frac{e^t}{e^t(1-t)+1}$

Example 2: Solve the IVP $ty' + y = t^4y^4, y(1) = 1/2$

ANS: $y = \frac{1}{t(11-3t)^{1/3}}$

Defn: A differential equation is called **homogeneous** if it can be written in the form $y' = f(\frac{y}{t})$.
That is, every occurrence of the variables on the right is in the form of a fraction y/t.
ANNOYING NOTE: Yes, this meaning of "homogeneous" is entirely different than the way we used it previously ("=0").

STEP 1: In this case, we **always** use $y_1 = t$. MEMORIZE IT!
STEP 2: Guess a solution of the form $y = uy_1$ (so $y = ut$).
STEP 3: Substitute into the original equation, rearranging to use $u = \frac{y}{t}$ when necessary. Then integrate to find u.

Example 2: $y' = \frac{y+te^{-y/t}}{t}$

Page Break

Bernoulli Equations

How to identify

A Bernoulli Equation has form:

$$y' + p(t)y = f(t)\cdot y^r$$

(r is any real number except $r=0$, $r=1$)

Steps to solve

1. Find y_1, a solution to complementary equation
2. Guess a solution $y = u\cdot y_1$
3. Substitute y, y' into original equation, solve for u', integrate to find u
   Then $y = u\cdot y_1$

Example 1: Solve the initial value problem $y' - y = ty^2$, $y(1)=e$

Bernoulli Eqn

$r=2$ (not $r=0$ or $r=1$)

Step1 complementary eq: $y' - y = 0$

$$+y \quad +y$$

$$\frac{y'}{y} = \frac{y}{y}$$

$$\int \frac{y'}{y}\,dt = \int 1\,dt$$

$$\ln|y| = t + C$$

$$e^{\ln|y|} = e^{t+C}$$

$$|y| = e^{t+C}$$

$$y = \pm e^{t+C}$$

choose $c=0$, "+" to get a single solution

$$\boxed{y_1 = e^t}\ *$$

STEP2 guess a solution $y = u\cdot y_1$

~ multiply y_1 by u to get the solution y

$$\boxed{y = ue^t}$$

$$\boxed{y' = ue^t + u'e^t}$$

STEP3 substitute into original eq: $y' - y = ty^2$

$$ue^t + u'e^t - ue^t = t(ue^t)^2$$

$$\frac{u'\cancel{e^t}}{\cancel{e^t}u^2} = \frac{tu^{\cancel{2}}e^{2t}}{e^t\cancel{u^2}}$$

$$\int \frac{u'}{u^2}\,dt = \int te^t\,dt$$

by parts

$u=t \quad dv = e^t\,dt$

$du = dt \quad v = e^t$

$$\int u^{-2}\cdot u'\,dt = te^t - \int e^t\,dt$$

$$\frac{u^{-1}}{-1} = te^t - e^t + C$$

$$u^{-1} = -te^t + e^t - C$$

$$\frac{1}{u} = -te^t + e^t - c$$

$$\boxed{u = \frac{1}{-te^t + e^t - c}}$$

solution $\quad y = uy_1$

$$y = \frac{1}{-te^t + e^t - c} \cdot e^t$$

$$\boxed{y = \frac{e^t}{-te^t + e^t - c}}$$

$y(1) = e$

substitute

$$e = \frac{e^1}{-1e^1 + e^1 - c}$$

$$e = \frac{e}{-c}$$

$$(-e)\ \frac{1}{e} = \frac{+c}{e}\ (-e)$$

$$\boxed{-1 = c}$$

$$y = \frac{e^t}{-te^t + e^t - (-1)}$$

$$\boxed{y = \frac{e^t}{-te^t + e^t + 1}}$$

particular solution satisfying $y(1) = e$.

Homogeneous "y over x" Equations

How to identify

if it can be written in the form $y' = f\left(\frac{y}{t}\right)$

"every occurence of the variables on the right is of the form $\frac{y}{t}$.

Steps to solve

1. we always use $y_1 = t$
2. Guess $y = u \cdot y_1$ $(y = ut)$
3. substitute. (rearrange to use $u = \frac{y}{t}$) solve for u', seperate variables, integrate to find u.

$e^{\frac{5y}{t}} = e^{5\left(\frac{y}{t}\right)}$

$e^{-\left(\frac{y}{t}\right)}$

Example 2: $y' = \frac{y+te^{-y/t}}{t}$

$y' = \frac{y}{t} + \frac{\cancel{t}e^{-y/t}}{\cancel{t}}$

$y' = \frac{y}{t} + e^{-y/t}$ ← this is a homogeneous y over t

STEP1: let $y_1 = t$

STEP2: guess $y = u \cdot y_1$, $\boxed{y = u \cdot t}$ → can rearrange to get $\boxed{u = \frac{y}{t}}$

$\boxed{y' = u + u't}$

STEP3: substitute into $y' = \frac{y}{t} + e^{-y/t}$

$$u + u't = \frac{u \cdot \cancel{t}}{\cancel{t}} + e^{-\frac{u \cdot \cancel{t}}{\cancel{t}}}$$

$$u + u't = u + e^{-u}$$

$$-u \qquad -u$$

$$\frac{u'\cancel{t}}{\cancel{t}e^{-u}} = \frac{\cancel{e^{-u}}}{t\,\cancel{e^{-u}}}$$

$$\frac{u'}{e^{-u}} = \frac{1}{t}$$

$$\int e^u \cdot u' \, dt = \int \frac{1}{t} \, dt$$

$$e^u = \ln|t| + c$$

$$\ln e^u = \ln(\ln|t| + c)$$

$$\boxed{u = \ln(\ln|t| + c)}$$

$$y = ut$$

$$\boxed{y = (\ln(\ln|t| + c))t}$$

general solution.

$$\boxed{y = t \cdot \ln\left(\ln|t| + c\right)}$$