Continued from last time

STRATEGY to solve non-homogeneous first-order linear differential equations.

STRATEGY to solve $y' + p(x)y = f(x)$:

STEP 1: Solve the complementary equation $y' + p(x)y = 0$, let $y_1(x)$ be a solution.

STEP 2: ~~Look for~~ "Guess" a solution of the form $y = uy_1$. Substitute into the original equation.

STEP 3: Solve for u by isolating u' and integrating.

STEP 4: Substitute $y_1(x)$, $u(x)$ to find the final solution: $y = uy_1$

First order linear nonhomogeneous.
$p(x) = 2$
$f(x) = x^3 e^{-2x}$

Example 3: $y' + 2y = x^3 e^{-2x}$

Step 1 complementary:

$y' + 2y = 0$ first order linear homogeneous

to solve, first separate variables

$-2y \quad -2y$

$\frac{y'}{y} = \frac{-2y}{y}$

$\int \frac{y'}{y}dx = \int -2\,dx$

$\ln|y| = -2x + C$

$e^{\ln|y|} = e^{-2x+C}$

$|y| = e^{-2x}e^{C}$

$y = \pm e^{C} e^{-2x}$ | this is the general solution to the complementary eqn.

we need a particular solution - choose C to be anything, choose $\pm$

choose +, choose $C=0$

$y_1 = e^{0} e^{-2x} = e^{-2x}$

$\boxed{y_1 = e^{-2x}}$ y_1 is a solution to the complementary equation.

STEP 2: guess a solution $y = u \cdot y_1$, u is an unknown function of x.

$\boxed{y = u \cdot e^{-2x}}$ guess

$\longrightarrow$ $y' + 2y = x^3 e^{-2x}$

substitute $y = u \cdot e^{-2x}$ take derivative:

$y' = u\left(e^{-2x} \cdot (-2)\right) + u' \cdot e^{-2x}$

$y' = -2ue^{-2x} + u'e^{-2x}$

sub: $-2ue^{-2x} + u'e^{-2x} + 2(ue^{-2x}) = x^3 e^{-2x}$

STEP 3: solve for u', integrate to find u.

$\cancel{-2ue^{-2x}} + u'e^{-2x} + \cancel{2ue^{-2x}} = x^3 e^{-2x}$

$\frac{u'\cancel{e^{-2x}}}{\cancel{e^{-2x}}} = \frac{x^3 \cancel{e^{-2x}}}{\cancel{e^{-2x}}}$

$\int u'dx = \int x^3 dx$

$u = \frac{x^4}{4} + C$ | general form of u.

STEP 4 $y = u \cdot y_1$

$\boxed{y = \left(\frac{x^4}{4} + C\right)e^{-2x}}$ || This is the general solution to the original differential equation.

or

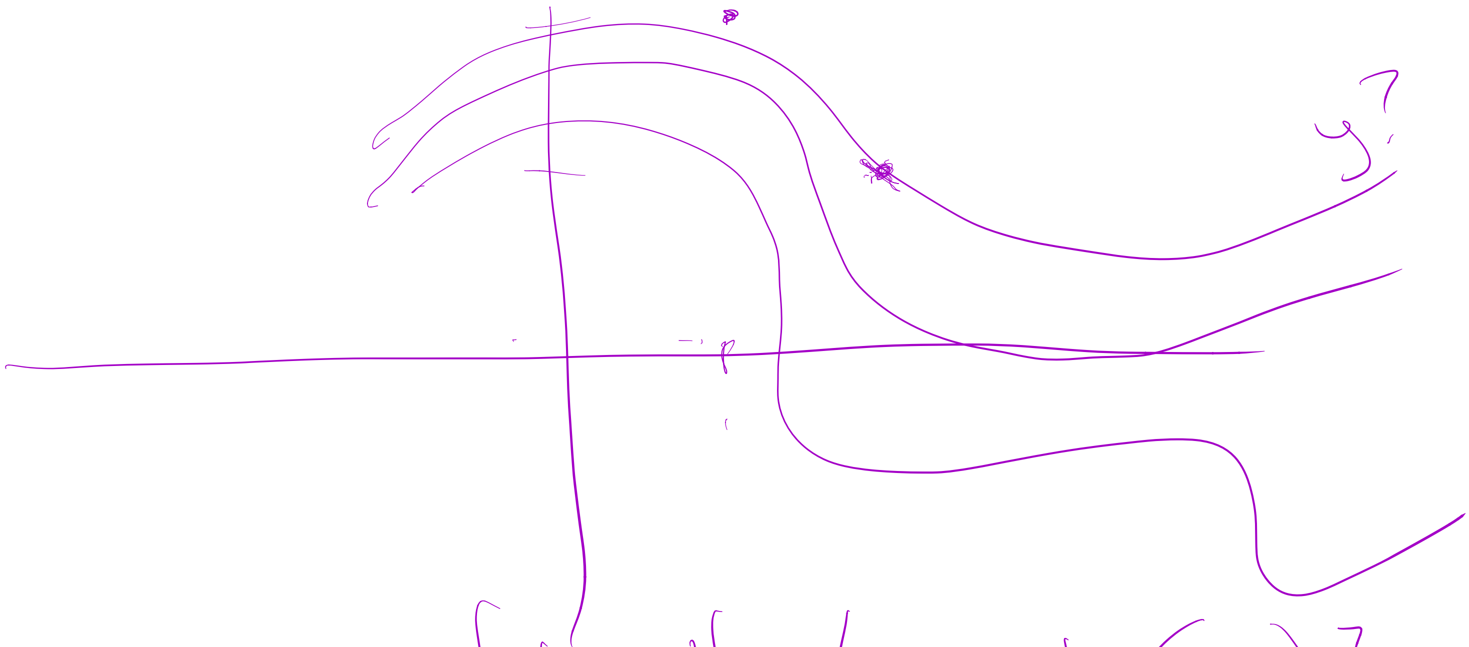
$\boxed{y = \frac{x^4}{4}e^{-2x} + Ce^{-2x}}$

Seperable

$y' = -\frac{x}{y}$

Goal: find the unknown function y.

tells us the slope of the function y at a point (x, y)

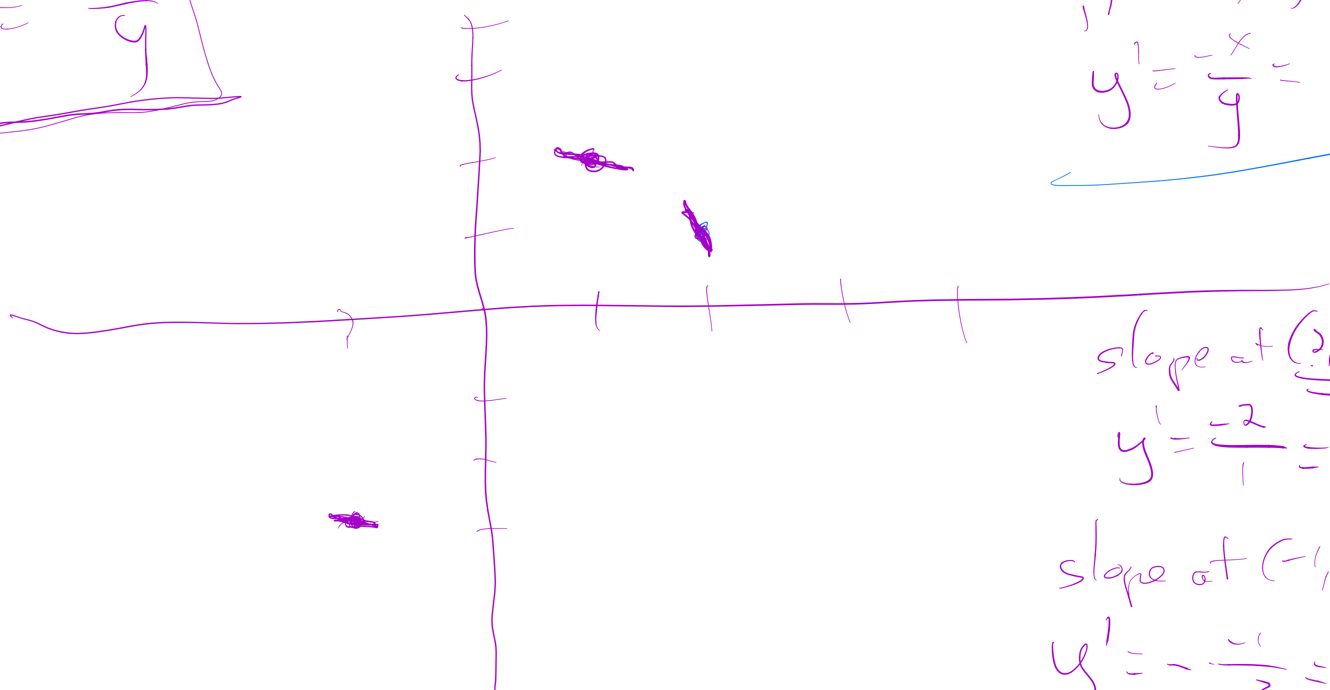


what is the slope at $(1,2)$?
if y passes through that point

$y' = -\frac{x}{y}$

if $x=1$, $y=2$

$y' = -\frac{x}{y} = -\frac{1}{2}$



slope at $(2,1)$?

$y' = \frac{-2}{1} = -2$

slope at $(-1,-3)$

$y' = -\frac{-1}{-3} = -\frac{1}{3}$

Defn a diffy Q is **separable** if it can be written: $h(y)\cdot y' = g(x)$

> To solve a separable equation,
> - separate variables
> - integrate both sides
> - solve for y.

Ex: $y' = -\frac{x}{y}$ Is it separable?

$\cdot y \qquad \cdot y$ multiply by y.

$$\int y\, y' dx = \int -x\, dx$$

$$\boxed{\frac{y^2}{2} = -\frac{x^2}{2} + C}$$

we call this an **implicit solution**.

solve for y to get an explicit solution

$$y^2 = -x^2 + 2C$$

$$\sqrt{y^2} = \pm\sqrt{-x^2+2C}$$

$$\boxed{\begin{aligned} y &= +\sqrt{-x^2+2C} \\ y &= -\sqrt{-x^2+2C} \end{aligned}}$$

find particular solution s.t.

$y(-2) = -3$

⋮

$c = 6.5$?

solution:

$y = -\sqrt{-x^2 + 2(6.5)}$

$$\boxed{y = -\sqrt{-x^2 + 13}}$$