# The Big Idea:

If $P(n)$ is <u>any</u> statement about the number $n$.
If we want to prove:

$$\forall n \in \mathbb{N}\, P(n)$$

It is enough to prove the following two thing:

① $P(1)$ ← base case, or base step

② $P(k) \to P(k+1)$ ← inductive step.

## Outline for Proof by induction

<u>Prop</u> $\forall n \in \mathbb{N}, P(n)$

<u>Proof</u> <u>(Induction)</u>:

<u>Base step</u> [prove $P(1)$ is true]

<u>Inductive step</u> [prove $P(k) \to P(k+1)$]

$\square$

## Example

**Prop.** $\forall n \in \mathbb{N}$, $4 \mid 5^n - 1$

**Proof (Induction).**

**Base step:** if $n=1$, we want to show $4 \mid 5^1 - 1$ Goal: $P(1)$

Consider $5^1 - 1 = 5 - 1 = 4$ and $4 \mid 4$. thus $4 \mid 5^1 - 1$ $\square$

**Inductive step:**

Goal: prove, for $k$ a natural number

$$(4 \mid 5^k - 1) \to (4 \mid 5^{k+1} - 1)$$

**Suppose** $4 \mid 5^k - 1$ $P(k)$

$$5^k - 1 = 4m, \text{ some } m \in \mathbb{Z}.$$

$$5 \cdot (5^k - 1) = 5 \cdot 4m$$

$$5 \cdot 5^k - 5 = 20m$$

$$5^{k+1} - 5 = 20m$$

$$+4 \qquad +4$$

$$5^{k+1} - 1 = 20m + 4$$

$$5^{k+1} - 1 = 4(5m+1)$$ note $p = 5m+1 \in \mathbb{Z}$

what is $P(n)$? what is being claimed about the natural number $n$?

$\boxed{P(n): 4 \mid 5^n - 1}$

if $n=7$,

$P(7)$ says "$4 \mid 5^7 - 1$"

$4 \mid 78125 - 1$

$4 \mid 78124$ ?

Yes

$4 \cdot 19531 = 78124$

Is $P(n)$ always true?

$n=2$

$P(2): 4 \mid 5^2 - 1$

$4 \mid 25 - 1$

$4 \mid 24$ True

$n=3$

$P(3): 4 \mid 5^3 - 1$

$5^{k+1} - 1 = 4p$, some $p \in \mathbb{Z}$

Thus $4 \mid 5^{k+1} - 1$ $\quad P(k+1)$

Thus by induction

$\forall n \in \mathbb{N}, 4 \mid 5^n - 1$ $\square$

$4 \mid 125 - 1$

$4 \mid 124$ true

If this $P(n)$: $\boxed{P(n): 4 \mid 5^n - 1}$

Q1: $\boxed{P(1): 4 \mid 5^1 - 1}$

$$P(k) \to P(k+1)$$

$$(4 \mid 5^k - 1) \to (4 \mid 5^{k+1} - 1)$$

$a \mid b$ means:

$b = am, m \in \mathbb{Z}$.

$2 \mid 10$

$10 = 2 \cdot 5$

<u>Example 2</u> $\quad \forall n \in \mathbb{N}, P(n)$

<u>Prop.</u> $\forall n \in \mathbb{N}, \; 5 \mid n^5 - n$

<u>Proof (Induction).</u>

Base case: if $n=1$, then $1^5 - 1 = 1 - 1 = 0$

test:

try the statement for

Since $0 \cdot 5 = 0$ and $0 \in \mathbb{Z}$, we have $5 \mid 0$ so $5 \mid 1^5 - 1$ $\square$

<u>Inductive case</u> Suppose for $k \in \mathbb{N}$

$$5 \mid k^5 - k$$

$$k^5 - k = 5 \cdot m, \; m \in \mathbb{Z}$$

$$+5k^4 + 10k^3 + 10k^2 + 5k \qquad +5k^4 + 10k^3 + 10k^2 + 5k$$

$$k^5 + 5k^4 + 10k^3 + 10k^2 + 4k = 5m + 5k^4 + 10k^3 + 10k^2 + 5k$$

$$k^5 + 5k^4 + 10k^3 + 10k^2 + 4k = 5(m + k^4 + 2k^3 + 2k^2 + k)$$

let $n = m + k^4 + 2k^3 + 2k^2 + k \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under $+, \times$

$$k^5 + 5k^4 + 10k^3 + 10k^2 + 4k = 5n, \; n \in \mathbb{Z}$$

$$k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 = 5n, \; n \in$$

$$(k+1)^5 - (k+1) = 5n, \; n \in \mathbb{Z}.$$

Thus $5 \mid (k+1)^5 - (k+1)$ by defn of divides

Thus, by induction, $\forall n \in \mathbb{N}$, $5 \mid n^5 - n$ $\square$

two natural numbers $n$.

$n = 2$

$2^5 - 2$

$32 - 2 = 30$ $\quad 5 \mid 30$

$P(2)$ is True.

$n = 3$

$3^5 - 3 = 243 - 3$

$= 240$

$P(3)$ is true. $5 \mid 240$

$P(n)$: $5 \mid n^5 - n$

$P(1)$: $5 \mid 1^5 - 1$

$P(k)$: $5 \mid k^5 - k$

$P(k+1)$: $5 \mid (k+1)^5 - (k+1)$

"$5 \mid 0$" means:

$0 = 5 \cdot m, \; m \in \mathbb{Z}$

$m = 0$ works!

$0 \mid 5 \leftarrow$ False.

$a|b$ means that $\frac{b}{a}$ is an integer.

Lets expand

$$(x+1)^5 = (x+1)(x+1)(x+1)(x+1)(x+1)$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
1 & 6 & & 15 & & 20 & & 15 & & 6 & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

$$= x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

$$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

**Example** for $n \in \mathbb{N}$, what is the sum of the first $n$ odd natural numbers?

**$n=3$** sum of the first 3 odd natural numbers:

$$1+3+5=9$$

**$n=4$** $1+3+5+7=16$

| $n$ | sum of first $n$ odd naturals |
|---|---|
| 1 | 1 |
| 2 | $1+3=4$ |
| 3 | 9 |
| 4 | 16 |
| 5 | $1+3+5+7+9=25$ |
| 6 | $1+3+5+7+9+11=36$ |

**~~Conjecture~~ Prop**: for any $n \in \mathbb{N}$, the sum of the first $n$-many odd natural numbers equals $n^2$.

$P(n)$: the sum of the first odd

n-many natural numbers is $n^2$

$P(1)$: the sum of the first 1-many odd natural numbers is $1^2$.

$P(n)$:

$P(k+1)$:

Using sigma notation:

the sum of the first n-many odd natural #s:

$$\boxed{\underline{\text{Prop:}} \quad \forall n \in \mathbb{N}, \quad \sum_{i=1}^{n} (2i-1) = n^2}$$

$i=1,\ 2\cdot 1-1=1$

$i=2,\ 2\cdot 2-1=3$

$i=3,\ 2\cdot 3-1=5$