Vocabulary

- open sentence - converse - if and only if	- logically equivalent - contrapositive - De Morgan's Laws

$x+7=2$

$\boxed{2(x+7)} = \boxed{2x+14}$

Definitions and Notation

- A sentence whose truth depends on the value of one or more variables is called an **open sentence**. An **open sentence** is not a statement.
- The statement $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$.
  NOTE: A conditional statement and its converse express entirely different things!
- $P \Leftrightarrow Q$ means $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. It is read "P if and only if Q".
- List of alternative phrases, all of which mean "$P \Leftrightarrow Q$"
  - P if and only if Q
  - P is a necessary and sufficient condition for Q.
  - For P is it necessary and sufficient that Q.
  - If P, then Q, and conversely.
- Two statements are **logically equivalent** if their truth values match up line-for-line in a truth table. In symbols, we express this using the equals sign.
- RULE: We are allowed to replace a statement with a logically equivalent statement
- The **contrapositive** of $P \Rightarrow Q$ is $(\sim Q) \Rightarrow (\sim P)$.

Example 1: Truth table for $P \Leftrightarrow Q$

P	Q	$P \Leftrightarrow Q$ $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	
T	F	
F	T	
F	F	

Common Logical Equivalences

- contrapositive: $P \Rightarrow Q = (\sim Q) \Rightarrow (\sim P)$
- De Morgans Laws: a) $\sim (P \wedge Q) = (\sim P) \vee (\sim Q)$
  b) $\sim (P \vee Q) = (\sim P) \wedge (\sim Q)$

Example 3

a) Write an expression that means "Either P or Q is true, but not both".
b) Complete the truth table for the expression below.

P	Q	$(P \vee Q) \wedge \sim (P \wedge Q)$
T	T	
T	F	
F	T	
F	F	

Example 4

Consider the statement
S: The product xy equals zero if and only if $x = 0$ or $y = 0$.
a) Express S in symbols as a combination of the three simple statements P, Q and R, where
P: $xy = 0$ Q: $x = 0$ R: $y = 0$
b) Write a truth table for the expression you created above.
HINT: start with columns for P, Q, and R, and consider all combinations of T and F for these three statements.

Example 5

Determine whether the statements $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are logically equivalent.

Logic

$x+3=5$

Statement - can be true or false

operators

$\wedge$ conjunction, "and"
$\vee$ disjunction, "or"
$\sim$ negation, "not"
$P \to Q$ conditional "if... then"

$P \leftrightarrow Q$ biconditional "if and only if"
shorthand for: $\boxed{P \to Q \text{ and } Q \to P}$

Example 1: Truth table for $P \Leftrightarrow Q$

P	Q	$P \Leftrightarrow Q$ $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T **T** T
T	F	F **F** T
F	T	T **F** F
F	F	T **T** T

Example:
"Dogs can bark if and only if pigs can fly"
Is this statement true?
false

"If I'm elected president then I will create universal health care"
- some politician

Question is

$$P \leftrightarrow Q = (P \wedge Q) \vee (\sim P \wedge \sim Q)$$

claim: these are logically equivalent

to show this is true, show the truth tables match up row for row.

We want to know if this is <u>true</u> (the politician kept his word) or <u>false</u> (the politician lied)

P	Q	$P \leftrightarrow Q$	$(P \wedge Q) \vee (\sim P \wedge \sim Q)$
T	T	T	T **T** F F F
T	F	F	F **F** F F T
F	T	F	F **F** T F F
F	F	T	F **T** T T T

Thus $P \leftrightarrow Q$ is logically equivalent to $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ ☑

Recall the conditional $P \rightarrow Q$

then $Q \to P$ is called the **converse**.

Ques: is the converse logically equivalent to the original conditional?

$$(P \to Q \equiv Q \to P)$$

P	Q	$P \to Q$	$Q \to P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Not logically equivalent

"they don't mean the same thing"

$P \to Q \neq Q \to P$

what is this equivalent to?

is $P \to Q$ equivalent to:

$Q \to P$ converse **NO**

$\sim P \to Q$

$P \to \sim Q$

$\sim P \to \sim Q$

$\boxed{\sim Q \to \sim P}$ contrapositive.

Fact $P \to Q$ is logically equiv. to the contrapositive $(\sim Q) \to (\sim P)$

$$\boxed{P \to Q \equiv (\sim Q) \to (\sim P)}$$

P	Q	$P \to Q$	$\sim Q \to \sim P$
T	T	T	F T F
T	F	F	T F F
F	T	T	F T T
F	F	T	T T T

Yes, logically equivalent!

Thus, we can always replace $P \to Q$ with the contrapositive $\sim Q \to \sim P$.

they mean the same thing!

Example 4

Consider the statement

S: The product xy equals zero if and only if $x = 0$ or $y = 0$.

a) Express S in symbols as a combination of the three simple statements P, Q and R, where

P: $xy = 0$ Q: $x = 0$ R: $y = 0$

b) Write a truth table for the expression you created above.

HINT: start with columns for P, Q, and R, and consider all combinations of T and F for these three statements.

$P \leftrightarrow (Q \vee R)$

P	Q	R	$P \leftrightarrow (Q \vee R)$		
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F	F	T	F

$x=0$
$y=0$
$xy=0$

$x=5$
$y=5$
$xy=25$

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1. Suppose $A_1 = \{a,b,d,e,g,f\}$, $A_2 = \{a,b,c,d\}$, $A_3 = \{b,d,a\}$ and $A_4 = \{a,b,h\}$.

(a) $\bigcup_{i=1}^{4} A_i =$ (b) $\bigcap_{i=1}^{4} A_i =$

2. Suppose $\begin{cases} A_1 = \{0,2,4,8,10,12,14,16,18,20,22,24\}, \\ A_2 = \{0,3,6,9,12,15,18,21,24\}, \\ A_3 = \{0,4,8,12,16,20,24\}. \end{cases}$

(a) $\bigcup_{i=1}^{3} A_i =$ (b) $\bigcap_{i=1}^{3} A_i =$

3. For each $n \in \mathbb{N}$, let $A_n = \{0,1,2,3,\ldots,n\}$.

(a) $\bigcup_{i \in \mathbb{N}} A_i =$ (b) $\bigcap_{i \in \mathbb{N}} A_i =$

4. For each $n \in \mathbb{N}$, let $A_n = \{-2n, 0, 2n\}$.

(a) $\bigcup_{i \in \mathbb{N}} A_i =$ (b) $\bigcap_{i \in \mathbb{N}} A_i =$

5. (a) $\bigcup_{i \in \mathbb{N}} [i, i+1] =$ (b) $\bigcap_{i \in \mathbb{N}} [i, i+1] =$

6. (a) $\bigcup_{i \in \mathbb{N}} [0, i+1] =$ (b) $\bigcap_{i \in \mathbb{N}} [0, i+1] =$

$\bigcup_{i \in \mathbb{N}} \{0,1,2,3,\ldots,i\}$

3. For each $n \in \mathbb{N}$, let $A_n = \{0,1,2,3,\ldots,n\}$.

(a) $\bigcup_{i \in \mathbb{N}} A_i = \{0,1,2,3,\ldots\}$ **(b)** $\bigcap_{i \in \mathbb{N}} A_i = \{0,1\}$

what can i equal?

$i=1$ $A_1 = \{0,1\}$

$i=2$ $A_2 = \{0,1,2\}$

$i=3$ $A_3 = \{0,1,2,3\}$

A_4

$\vdots$ $\vdots$

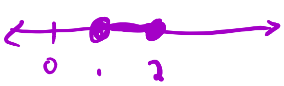
$i=100$ $A_{100} = \{0,1,2,3,\ldots,100\}$

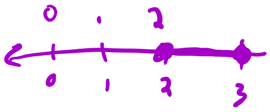
$\vdots$

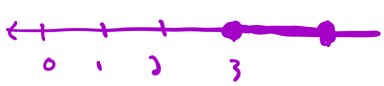
5. **(a)** $\bigcup_{i \in \mathbb{N}} [i, i+1] =$ **(b)** $\bigcap_{i \in \mathbb{N}} [i, i+1] =$

$A_i :=$ the closed interval $[i, i+1]$, $A_i = [i, i+1]$

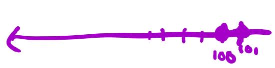
$\bigcup_{i \in \mathbb{N}} A_i$

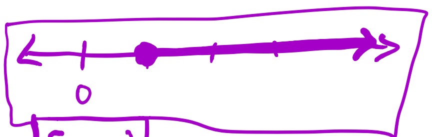
$i=1$, $A_1 = [1,2]$ 

$i=2$, $A_2 = [2,3]$ 

$A_3 = [3,4]$ 

$\vdots$

$A_{100} = [100, 101]$ 

$\bigcup_{i \in \mathbb{N}} [i, i+1] =$ 

$= \boxed{[1, \infty)}$

$\bigcap [i, i+1] = \{\ \} = \varnothing$

$$\bigcap_{i\in\mathbb{N}} \left(0, \tfrac{1}{i}\right] = \{\ \} = \emptyset$$