

## Vocabulary

- ordered pair  
- Cartesian product  
- ordered triple

- Cartesian power  
- subset

### Definitions and Theorems

- An **ordered pair** is a list  $(x,y)$  of two things,  $x$  and  $y$ , enclosed in parentheses and separated by a comma. *NOTE: unlike a set, the **order** of the elements is important:*  
 $(2,4)$  is *NOT* the same as  $(4,2)$ .
- The **Cartesian product** of two sets  $A$  and  $B$  is another set, written  $A \times B$ , and defined as  
 $A \times B = \{(a,b) : a \in A, b \in B\}$ .
- Theorem. If  $A$  and  $B$  are finite sets,  $|A \times B| = |A| \times |B|$ .
- An **ordered triple** is a list  $(x,y,z)$ .
- A **Cartesian power**, like  $\mathbb{R}^2$ , is simply shorthand for the product of a set with itself  
 $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$  (similar for higher powers:  $\mathbb{N}^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ )

**Example 1:** If  $A = \{p,q,r\}$  and  $B = \{w,x\}$ , find  $A \times B$ .

**Example 2:** i) Describe the Cartesian product  $\mathbb{R} \times \mathbb{R}$ .

ii) If  $A$  is the closed interval  $[0,1]$  and  $B$  is the half-open interval  $[2,3)$ , draw a sketch of  $A \times B$ .

**Example 3:** If  $A = \{3,7\}$ ,  $B = \{2,4\}$ , and  $C = \{5,9\}$ , then:

i) is  $(3,2,9) \in A \times B \times C$ ?

ii) is  $(3,5,2) \in A \times B \times C$ ?

### Definition

If  $A$  and  $B$  are sets and every element of  $A$  is also an element of  $B$ , then we say  **$A$  is a subset of  $B$**  and we write  $A \subseteq B$ . If this is *NOT* the case then we say  **$A$  is not a subset of  $B$** , and we write  $A \not\subseteq B$ .

*NOTE:  $A \not\subseteq B$  means there is at least one element of  $A$  that is not an element of  $B$ .*

member  $x \in A$  element of  
can be anything must be a set

$A \subseteq B$   
must both be sets

#### Example 4

If  $A = \{2, 3, 5\}$ ,  $B = \{2, 3, 4, 5, 6, 7, 8\}$  and  $C = \{1, 2, 3\}$

- i) is  $A \subseteq B$ ? Why?
- ii) is  $A \subseteq C$ ? Why?
- iii) is  $C \subseteq A$ ? Why?
- iv) is  $A \subseteq A$ ? Why?
- v) is  $\emptyset \subseteq A$ ? Why?

Theorem: Every set is a subset of itself,  $A \subseteq A$

Theorem: The empty set is a subset of every set: for any set  $A$ ,  $\emptyset \subseteq A$

#### Example 5

a) If  $A = \{\pi, 5\}$  and  $B = \{4, 7\}$ , then

- i) Find  $A \times B$  and  $B^2$
- ii) is  $(\pi, 7) \in A \times B$ ?
- iii) is  $(4, 5) \in B \times A$ ?
- iv) is  $(\pi, \pi) \in A^2$ ?

b) If  $A = \{\{4, 5, 6\}, \emptyset\}$  and  $B = \{\mathbb{N}, \mathbb{Z}, (\emptyset, \{2, 7\})\}$ , then

- i) is  $(\{4, 5, 6\}, \mathbb{Z}) \in A \times B$ ?
- ii) is  $(\emptyset, \emptyset) \in A \times B$ ?
- iii) Find  $A \times B$ . What is  $|A \times B|$ ?
- iv) is  $(\emptyset, \{2, 7\}), \mathbb{N}, \{4, 5, 6\} \in B^3$ ? What product of A's and B's is it an element of?

c) Sketch each set in the plane.

- i)  $[1, 2] \times (3, 5)$
- ii)  $(-1, -\frac{1}{2}] \times [2, 3)$
- iii)  $[0, 1] \times \{1, 3, 5\}$

d) Consider the set with two elements  $\{5, \{5\}\}$ . True or False:

- i)  $5 \in \{5, \{5\}\}$
- ii)  $5 \subseteq \{5, \{5\}\}$
- iii)  $\{5\} \in \{5, \{5\}\}$
- iv)  $\{5\} \subseteq \{5, \{5\}\}$
- v)  $\{\{5\}\} \in \{5, \{5\}\}$
- vi)  $\{\{5\}\} \subseteq \{5, \{5\}\}$

e) True or False:

- i)  $\{(1, 1), (2, 6), (5, -1), (3, 2)\} \subseteq \mathbb{Z} \times \mathbb{Z}$
- ii)  $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{R} \times \mathbb{R}$

Example 1: If  $A = \{p, q, r\}$  and  $B = \{w, x\}$ , find  $A \times B$ .

$$A \times B = \{(p, w), (q, w), (r, w), (p, x), (q, x), (r, x)\}$$

Cartesian product of A and B

$$|A| = 3$$

$$|B| = 2$$

$$|A \times B| = 6$$

$$|A \times B| = |A| \times |B|$$

Example 2: (i) Describe the Cartesian product  $\mathbb{R} \times \mathbb{R}$ .

ii) If A is the closed interval  $[0, 1]$  and B is the half-open interval  $[2, 3)$ , draw a sketch of  $A \times B$ .

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 =$$

product of sets  $\mathbb{R} = \{x : x \text{ is a real number}\} = \text{all numbers on the number line}$

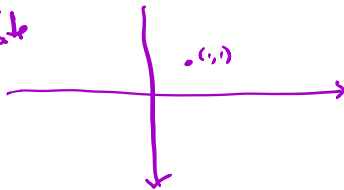
$$\mathbb{R} \times \mathbb{R} = \{(1, 2), (-1, -2), (-1.5, 5.7), \left(\frac{\pi}{2}, 0\right), (5, 0), (0, 0), \dots\}$$

undefined (not real number)

$$|\mathbb{R} \times \mathbb{R}| = \infty$$

$\{\infty, \psi\}$

$\mathbb{R} \times \mathbb{R} =$  picture: "the coordinate plane"



$(1, 1)$

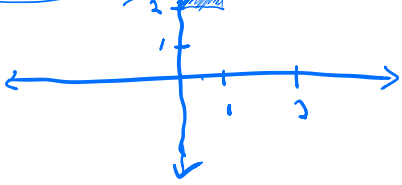
ii)  $A = [0, 1] =$   $A = \{0, 1, 1.5, 2.7, 3, \dots\}$   
 $|A| = \infty$

$B = [2, 3) =$   $B = \{2, 2.4, 2.9, \dots\}$   
 $|B| = \infty$

$$A \times B = \{(2.5, 2.9), (1, 2.9), (0.8, 2.5), \dots\}$$

$$|A \times B| = \infty$$

$(2.5, 2.9)$



**Example 3:** If  $A = \{3, 7\}$ ,  $B = \{2, 4\}$ , and  $C = \{5, 9\}$ , then:

- i) is  $(3, 2, 9) \in A \times B \times C$ ?  
 ii) is  $(3, 5, 2) \in A \times B \times C$ ? *No since  $5 \notin B, 2 \notin C$ .*

$$A \times B \times C$$

$$(3, 4, 5) \in A \times B \times C$$

**Example 4**

If  $A = \{2, 3, 5\}$ ,  $B = \{2, 3, 4, 5, 6, 7, 8\}$  and  $C = \{1, 2, 3\}$

- i) is  $A \subseteq B$ ? Why? *Yes*  
 ii) is  $A \subseteq C$ ? Why? *No*  $A \not\subseteq C$  since  $5 \in A$  but  $5 \notin C$ .  *$B \not\subseteq A$  since  $7 \in B, 7 \notin A$*   
 iii) is  $C \subseteq A$ ? Why? *No*  $C \not\subseteq A$  since  $1 \in C$  but  $1 \notin A$ .  
 iv) is  $A \subseteq A$ ? Why? *Yes*  
 v) is  $\emptyset \subseteq A$ ? Why? *Yes*

$\mathbb{R}, \emptyset$  means empty set  $\emptyset = \{ \}$

if  $\emptyset \not\subseteq A$  then we could find some element  $x \in \emptyset$  with  $x \notin A$ .

is  $A \subseteq \emptyset$ ? *No* since  $2 \in A$ , but  $2 \notin \emptyset$   
 so  $A \not\subseteq \emptyset$

### Example 5

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i) Find  $A \times B$  and  $B^2$

ii) is  $(\pi, 7) \in A \times B$ ?

iii) is  $(4, 5) \in B \times A$ ?

iv) is  $(\pi, \pi) \in A^2$ ?  $\sim A \times A$

b) If  $A = \{\{4, 5, 6\}, \emptyset\}$  and  $B = \{\mathbb{N}, \mathbb{Z}, (\emptyset, \{2, 7\})\}$ , then

i) is  $(\{4, 5, 6\}, \mathbb{Z}) \in A \times B$ ? Yes

ii) is  $(\emptyset, \emptyset) \in A \times B$ ? No, since  $\emptyset \notin B$ .

iii) Find  $A \times B$ . What is  $|A \times B|$ ?

iv) is  $(\emptyset, \{2, 7\}), \mathbb{N}, \{4, 5, 6\} \in B^3$ ? What product of A's and B's is it an element of?

$|A| = 2$   
 $4 \in A$ ? NO

$|B| = 3$

c) Sketch each set in the plane.

i)  $[1, 2] \times (3, 5)$

ii)  $(-1, -\frac{1}{2}] \times [2, 3)$

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i)  $5 \in \{5, \{5\}\}$

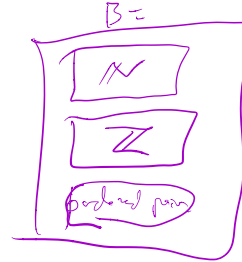
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v)  $\{\{5\}\} \in \{5, \{5\}\}$

vi)  $\{\{5\}\} \subseteq \{5, \{5\}\}$



e) True or False:

i)  $\{(1, 1), (2, 6), (5, -1), (3, 2)\} \subseteq \mathbb{Z} \times \mathbb{Z}$

ii)  $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{R} \times \mathbb{R}$