

# Geometric Series - Introduction to Geometric Sequences

$$5, 20, 80, 320, 1280, \dots$$

common ratio  $r=4$

Geometric Sequence  $\{a_n\}$   
Each two consecutive terms have a common ratio  $r$

$$a_n = a_{n-1} \cdot r$$

$$a_n = a_1 \cdot r^{n-1}$$

Recall: Arithmetic Sequence  $\{a_n\}$   
common difference  $d$

$$a_n = a_{n-1} + d$$

$$a_n = a_1 + d(n-1)$$

**Example 24.2.** Determine if the sequence is a geometric or arithmetic sequence, or neither or both. If it is a geometric or arithmetic sequence, then find the general formula.

- a) 3, 6, 12, 24, 48, ... Geometric Sequence  $a_1=3, r=2$
- b) 100, 50, 25, 12.5, ... Geometric  $a_1=100, r=\frac{1}{2}$
- c) 700, -70, 7, -0.7, 0.07, ... Geometric  $a_1=700, r=-\frac{1}{10}$
- d) 2, 4, 16, 256, ... Not Arith., Not Geo.
- e) 3, 10, 17, 24, ... Arithmetic  $a_1=3, d=7$
- f) -3, -3, -3, -3, -3, ...
- g)  $a_n = (\frac{3}{7})^n$
- h)  $a_n = n^2$

a)  $a_n = 3 \cdot 2^{n-1}$

b)  $a_n = 100 \cdot (\frac{1}{2})^{n-1}$

c)  $a_n = 700 \cdot (-\frac{1}{10})^{n-1}$

e)  $a_n = 3 + 7(n-1)$

Is e) Geometric? (common ratios:  
NOT Geometric  $\frac{10}{3} = 3.33...$   
 $\frac{17}{10} = 1.7$

f) -3, -3, -3, -3, -3...  
Geometric:  $\frac{-3}{-3} = 1, \frac{-3}{-3} = 1, r=1$  Yes it's Geometric  
 $a_n = a_1 \cdot r^{n-1} = -3 \cdot 1^{n-1} = -3$   
 $a_n = -3$

Arithmetic: common difference  $d=0$   
 $a_n = a_1 + d(n-1) = -3 + 0(n-1) = -3$   
 $a_n = -3$

g)  $a_n = (\frac{3}{7})^n$   
 $a_1 = (\frac{3}{7})^1 = \frac{3}{7}$   
 $a_2 = (\frac{3}{7})^2 = \frac{9}{49}$   
 $a_3 = (\frac{3}{7})^3 = \frac{27}{343}$   
 $a_4 = (\frac{3}{7})^4 = \frac{81}{2401}$

Not arithmetic  
common ratio  $r = \frac{3}{7}$   
Geometric  
 $a_n = a_1 \cdot r^{n-1} = (\frac{3}{7}) \cdot (\frac{3}{7})^{n-1} = (\frac{3}{7})^n$   
 $a_n = (\frac{3}{7})^n$

h)  $a_n = n^2$   
 $a_1 = 1^2 = 1$   
 $a_2 = 2^2 = 4$   
 $a_3 = 3^2 = 9$

Is it arithmetic? NO  
 $4-1=3$   
 $9-4=5$   
 $16-9=7$   
 not equal

Is it Geometric? NO

$$a_4 = 4 = 16$$

$$\frac{4}{1} = 4$$
$$\frac{9}{4} = 2.25$$

not equal

Finding  
the Formula  
for a  
Geometric  
Sequence -  
Example 24.3

$$a_n = a_1 \cdot r^{n-1}$$

Example 24.3. Find the general formula of a geometric sequence

with the given property

a)  $r = 4$ , and  $a_5 = 6400$

b)  $a_1 = \frac{2}{5}$ , and  $a_4 = -\frac{27}{20}$

c)  $a_5 = 216$ ,  $a_7 = 24$ , and  $r$  is positive

a)  $r = 4$ ,  $a_5 = 6400$   
 substitute:  $6400 = a_1 \cdot 4^{5-1}$   
 $6400 = a_1 \cdot 4^4$   
 $6400 = a_1 \cdot 256$   
 $\frac{6400}{256} = \frac{a_1 \cdot 256}{256}$   
 $25 = a_1$   
 $a_n = 25 \cdot 4^{n-1}$  ANS

$a_n = a_1 \cdot r^{n-1}$   
 b)  $a_1 = \frac{2}{5}$ ,  $a_4 = -\frac{27}{20}$   
 substitute:  
 $-\frac{27}{20} = \frac{2}{5} \cdot r^{4-1}$   
 $\frac{5}{2} \cdot -\frac{27}{20} = \frac{2}{5} \cdot r^3 \cdot \frac{5}{2}$   
 $-\frac{27}{8} = r^3$   
 $\sqrt[3]{-\frac{27}{8}} = \sqrt[3]{r^3}$   
 $\frac{\sqrt[3]{-27}}{\sqrt[3]{8}} = r$   
 $\frac{-3}{2} = r$   
 $a_n = \frac{2}{5} \cdot \left(\frac{-3}{2}\right)^{n-1}$

c)  $a_5 = 216$ ,  $a_7 = 24$ , and  $r$  is positive  
 $a_n = a_1 \cdot r^{n-1}$   
 $216 = a_1 \cdot r^{5-1}$        $24 = a_1 \cdot r^{7-1}$   
 $216 = a_1 \cdot r^4$        $24 = a_1 \cdot r^6$   
 $\frac{216}{r^4} = \frac{a_1 \cdot r^4}{r^4}$        $24 = \frac{216}{r^4} \cdot r^6$   
 $a_1 = \frac{216}{r^4}$        $24 = \frac{216}{r^4} \cdot r^6$   
 $\frac{24}{9 \cdot 216} = \frac{216 \cdot r^2}{216}$   
 $\frac{1}{9} = r^2$   
 $\pm \sqrt{\frac{1}{9}} = \sqrt{r^2}$   
 $\pm \frac{\sqrt{1}}{\sqrt{9}} = r$   
 $r = \pm \frac{1}{3}$

$r$  is positive, so  
 $r = \frac{1}{3}$   
 sub  $a_1 = \frac{216}{\left(\frac{1}{3}\right)^4} = \frac{216}{\frac{1}{81}} = \frac{216}{1} \cdot \frac{81}{1} = 216 \cdot 81$   
 $a_1 = 17496$   
 $a_n = 17496 \left(\frac{1}{3}\right)^{n-1}$

# the Sum of a Geometric Series - Introduction formula

Example 24.4. Consider the geometric sequence  $a_n = 8 \cdot 5^{n-1}$ , that is the sequence:

8, 40, 200, 1000, 5000, 25000, 125000, ...  
 $a_1 = 8$   
 $r = 5$

Find the sum of the first 6 terms of this sequence  $n = 6$

$$8 + 40 + 200 + 1000 + 5000 + 25000 = 31248$$

Goal:  $8 + 40 + 200 + 1000 + 5000 + 25000 = A$

$-40 - 200 - 1000 - 5000 - 25000 - 125000 = -5A$

Telescoping series

Add:

$8 - 125000$

$a_1 - a_7$

$-124992$

$-4$

$= -4A$

$= -4A$

$-4$

$(1-r)$

$a_1 - a_{k+1}$

$31248 = A$

$a_{k+1} = a_1 \cdot r^{(k+1)-1}$

$a_{k+1} = a_1 \cdot r^k$

sub

## Formula for finite sum of geometric series:

$\{a_n\}$  geometric series,  $a_n = a_1 \cdot r^{n-1}$

Find the sum of first  $n$  terms.

$$\sum_{i=1}^n a_i = a_1 \cdot \frac{1-r^n}{1-r}$$

last step: divide by  $1-r$ .

$\frac{a_1 - a_{k+1}}{1-r}$

$= \frac{a_1 - a_1 r^k}{1-r}$

$= \frac{a_1 (1-r^k)}{1-r}$

$= a_1 \cdot \frac{1-r^n}{1-r}$

Finding the finite sum of a geometric series - Example 24.6

Example 24.6. Find the value of the geometric series.

a) Find the sum  $\sum_{n=1}^6 a_n$  for the geometric sequence  $a_n = 10 \cdot 3^{n-1}$

b) Determine the value of the geometric series:  $\sum_{k=1}^5 \left(-\frac{1}{2}\right)^k$

c) Find the sum of the first 12 terms of the geometric sequence

$$-3, -6, -12, -24, \dots$$

a)  $a_1 + a_2 + a_3 + a_4 + a_5 + a_6$

$$a_1 = 10, r = 3, n = 6$$

$$\sum_{i=1}^6 10 \cdot 3^{i-1} = a_1 \frac{1-r^n}{1-r} = 10 \cdot \frac{1-3^6}{1-3} = 10 \cdot \frac{1-729}{-2} = 10 \cdot \frac{-728}{-2} = 10 \cdot 364 = 3640 \text{ ANS}$$

b)  $\sum_{k=1}^5 \left(-\frac{1}{2}\right)^k = -\frac{1}{2} \cdot \frac{1 - \left(-\frac{1}{2}\right)^5}{1 - \left(-\frac{1}{2}\right)} = -\frac{1}{2} \cdot \frac{1 - \frac{(-1)^5}{2^5}}{1 + \frac{1}{2}} = -\frac{1}{2} \cdot \frac{1 - \frac{-1}{32}}{\frac{3}{2}} = -\frac{1}{2} \cdot \frac{\frac{32}{32} + \frac{1}{32}}{\frac{3}{2}}$

$$a_1 = \left(-\frac{1}{2}\right)^1 = -\frac{1}{2}$$

$$r = -\frac{1}{2}$$

# of terms  $n = 5$

$$= -\frac{1}{2} \cdot \frac{\frac{33}{32}}{\frac{3}{2}}$$

$$= -\frac{1}{2} \cdot \frac{33}{32} \cdot \frac{2}{3} = -\frac{11}{32} \text{ ANS}$$

c)  $-3, -6, -12, -24, \dots$

sum of first 12 terms

$$a_1 = -3, r = 2, n = 12$$

$$\sum_{i=1}^{12} a_i = -3 \cdot \frac{1-2^{12}}{1-2} = -3 \cdot \frac{1-4096}{-1} = -3 \cdot \frac{-4095}{-1} = -3 \cdot 4095$$

$$= -12285 \text{ ANS}$$

Infinite  
Geometric  
Series  
Introduction  
+  
Formula

Example 24.7. Consider the geometric sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$

$r = \frac{1}{2}$   
 $a_1 = 1$

formula for  $a_n$ :

$$a_n = 1 \left(\frac{1}{2}\right)^{n-1}$$

$$a_n = \left(\frac{1}{2}\right)^{n-1}$$

What is the initial term? What is the common ratio?

Let's try adding up some of the terms. Try this by hand, and by using the formula for finite geometric series. What happens if we add up ALL the terms?

Partial sums

$n=1$        $1 = 1$

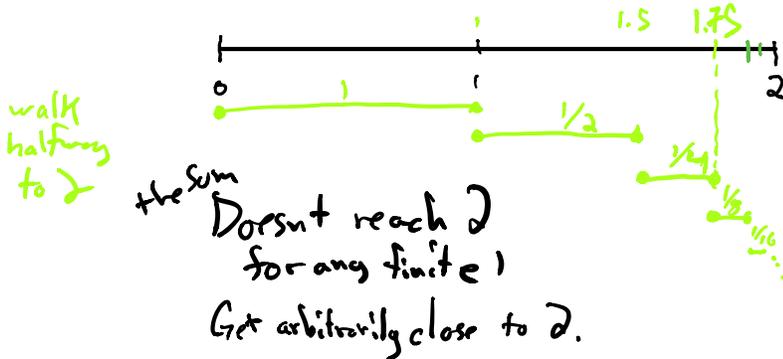
$n=2$        $1 + \frac{1}{2} = \frac{3}{2} = 1.5$

$n=3$        $1 + \frac{1}{2} + \frac{1}{4} = 1.75$

$n=4$        $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1.875$

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1.9375$

Question: are the partial sums approaching 2?



Recall

$$\sum_{i=1}^n a_i = a_1 \cdot \frac{1-r^n}{1-r} = 1 \cdot \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$$

as  $n \rightarrow \infty$ ,  
 $\left(\frac{1}{2}\right)^n \rightarrow 0$

as  $n \rightarrow \infty$ ,  $\left(\frac{1}{2}\right)^n \rightarrow 0$

$$\sum_{i=1}^{\infty} a_i = 1 \cdot \frac{1-0}{1-\frac{1}{2}} = 1 \cdot \frac{1}{\frac{1}{2}} = 1 \cdot 2 = \boxed{2}$$

Sum of an infinite geometric Series

$\{a_n\}$  geometric,  $a_n = a_1 \cdot r^{n-1}$ ,  $-1 < r < 1$

then 
$$\sum_{i=1}^{\infty} a_i = a_1 \cdot \frac{1}{1-r}$$

# Finding the Sum of an Infinite Geometric Series

Examples 24.10 + 24.11

**Example 24.10.** Find the value of the infinite geometric series.

- a)  $\sum_{j=1}^{\infty} a_j$ , for  $a_j = 5 \cdot \left(\frac{1}{3}\right)^{j-1}$
- b)  $\sum_{n=1}^{\infty} 3 \cdot (0.71)^n$
- c)  $500 - 100 + 20 - 4 + \dots$
- d)  $3 + 6 + 12 + 24 + 48 + \dots$

$$\sum_{i=1}^{\infty} a_i = a_1 \cdot \frac{1}{1-r}$$

a)  $\sum_{j=1}^{\infty} a_j$ ,  $a_j = 5 \cdot \left(\frac{1}{3}\right)^{j-1}$   
 $a_1 = 5$   
 $r = \frac{1}{3}$

$$\frac{\infty}{1} = 5 \cdot \frac{1}{1 - \frac{1}{3}} = 5 \cdot \frac{3}{2}$$

$$\sum_{j=1}^{\infty} a_j = 5 \cdot \frac{1}{1-\frac{1}{3}} = 5 \cdot \frac{3}{3-\frac{1}{3}} = 5 \cdot \frac{2}{3} = \frac{10}{3} = \frac{15}{2}$$

b)  $\sum_{n=1}^{\infty} 3 \cdot (0.71)^n$        $a_1 = 3 \cdot (0.71)^1 = 2.13$   
 $r = 0.71$

$$S = 2.13 \cdot \frac{1}{1-0.71} = \frac{2.13}{0.29} = \frac{213}{29} \approx 7.3448$$

c)  $500 - 100 + 20 - 4 + \dots$        $r = -\frac{1}{5}$       Is  $r$  between  $-1$  and  $1$ ?  
 $a_1 = 500$       Yes - can use formula

$$\sum_{i=1}^{\infty} a_i = 500 \cdot \frac{1}{1-(-\frac{1}{5})} = 500 \cdot \frac{1}{1+\frac{1}{5}} = 500 \cdot \frac{1}{\frac{6}{5}} = 500 \cdot \frac{5}{6} = 500 \cdot \frac{5}{6} = \frac{2500}{6} = \frac{1250}{3} \text{ ANS}$$

d)  $3 + 6 + 12 + 24 + 48 + \dots$       Geometric Series  
 $r = 2$   
 $a_1 = 3$   
 $r$  is not between  $-1$  and  $1$   
 Can't use formula.  
 $= \infty$

Example 24.11. Consider the real number given by  $0.555555\dots$   
 Rewrite this number as an infinite geometric series. Can you figure out what fraction it is equal to?

$$0.555555\dots$$

$$\begin{aligned}
 &= 0.5 \\
 &+ 0.05 \\
 &+ 0.005 \\
 &+ 0.0005 \\
 &+ 0.00005 \\
 &\vdots
 \end{aligned}$$

Is it geometric? Yes  
 $r = \frac{1}{10}$  Note:  $-1 < r < 1$  ✓  
 $a_1 = 0.5$

$$\boxed{0.55555\dots} = \sum_{i=1}^{\infty} a_i = 0.5 \cdot \frac{1}{1 - \frac{1}{10}} = \frac{1}{2} \cdot \frac{1}{\frac{10}{10} - \frac{1}{10}} = \frac{1}{2} \cdot \frac{1}{\frac{9}{10}} = \frac{1}{2} \cdot \frac{10}{9} = \boxed{\frac{5}{9}}$$

Challenge: What fraction is  
 $0.13131313\dots$   
 equal to?