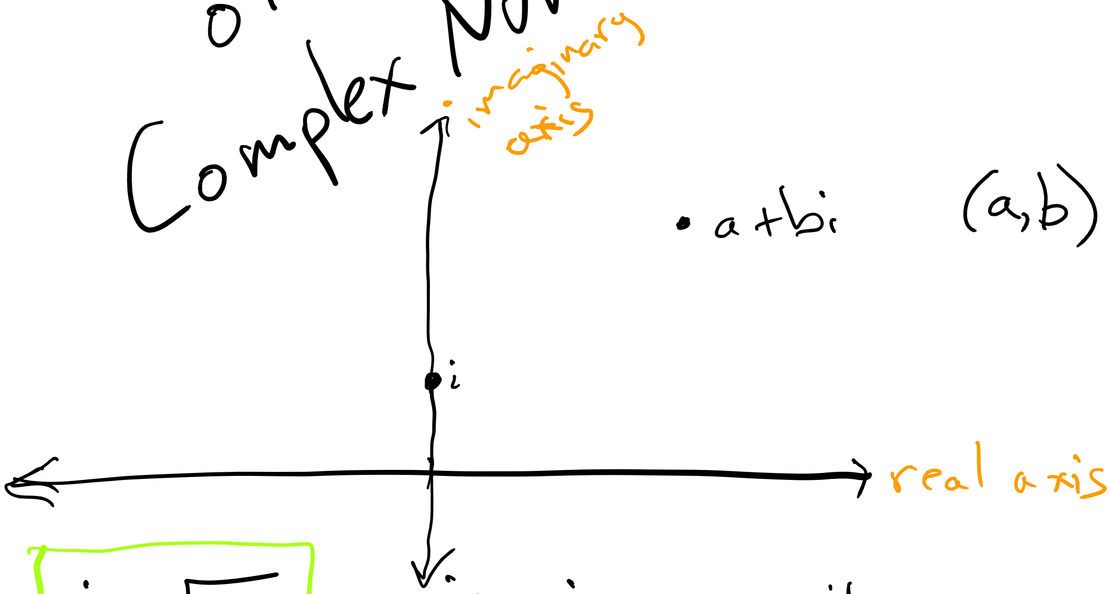
Review of Complex Numbers



Let $\boxed{\begin{array}{l} i=\sqrt{-1} \\ i^2=-1 \end{array}}$

Complex numbers: $a+bi$

real part

imaginary part

a, b are real numbers

$i=\sqrt{-1}$

$\mathbb{C}$ - complex numbers

$\mathbb{R}$ - real numbers

Example 21.3. Perform the operation.

a) $(2-3i)+(-6+4i)$

b) $(3+5i)\cdot(-7+i)$

c) $\frac{5+4i}{3+2i}$

$a+bi$

a) $(2-3i)+(-6+4i)=\boxed{-4+i}$

b) $(3+5i)\cdot(-7+i) = -21+3i-35i+5i^2$

$= -21-32i+5(-1)$

$= \boxed{-26-32i}$

c) $\dfrac{(5+4i)}{(3+2i)}\dfrac{(3-2i)}{(3-2i)} = a+bi\,?$

Strategy:
multiply top and bottom
by conjugate of bottom

$$= \frac{15-10i+12i-8i^2}{9\cancel{-6i+6i}-4i^2}$$

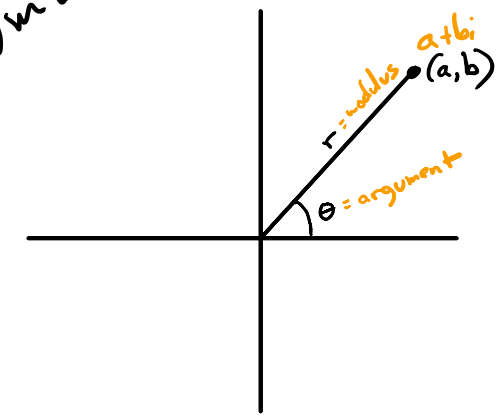
$$= \frac{15+2i-8(-1)}{9-4(-1)} = \frac{15+2i+8}{9+4}$$

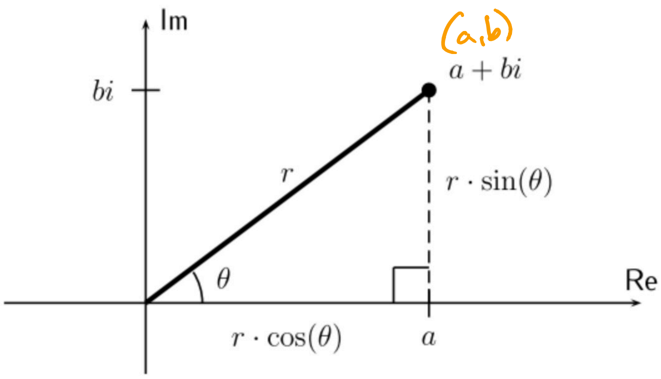
$$= \frac{23+2i}{13} \qquad a+bi$$

$$= \frac{23}{13}+\frac{2i}{13}$$

$$= \boxed{\frac{23}{13}+\frac{2}{13}i}$$

Polar Form of Complex Numbers





$a = r \cdot \cos\theta$

$b = r \cdot \sin\theta$

modulus

argument

$$\underbrace{\boxed{a+bi}}_{\text{standard form}} = r\cos\theta + r\sin\theta \cdot i = \underbrace{\boxed{r(\cos\theta + i\sin\theta)}}_{\text{polar form}}$$

Converting from standard form to polar form:

$r = \sqrt{a^2+b^2}$

$\tan(\theta) = \frac{b}{a}$

$\theta \stackrel{?}{=} \arctan\left(\frac{b}{a}\right)$ in Quadrants I, IV

$\theta = \arctan\left(\frac{b}{a}\right) + \pi$ in Quadrants II, III

$\theta = \arctan\left(\frac{b}{a}\right) + 180^\circ$

Example 21.7. Convert the complex number to polar form.

a) $2 + 3i$

b) $-2 - 2\sqrt{3}i$

c) $4 - 3i$

d) $-4i$

$r(\cos\theta + i\sin\theta)$

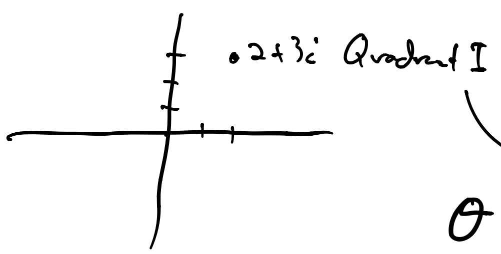
a) $2 + 3i$

find modulus $r = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$

$\tan^{-1}(1.5)$

find θ $\arctan\left(\frac{3}{2}\right) = \arctan(1.5) = 56.3^\circ$ Quadrant I

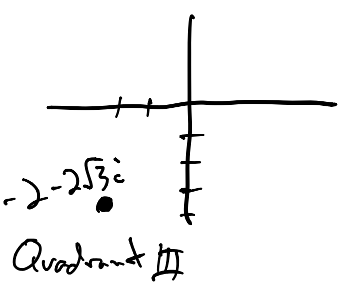
DEGREES



$\theta \approx 56.3^\circ$

$2 + 3i = \boxed{\sqrt{13}(\cos 56.3^\circ + i \sin 56.3^\circ)}$

b) $-2 - 2\sqrt{3}i$



modulus: $r = \sqrt{(-2)^2 + (-2\sqrt{3})^2}$

$= \sqrt{4 + 4\cdot 3}$

$= \sqrt{4 + 12} = \sqrt{16} = 4$ $r = 4$

argument: $\tan(\theta) = \frac{\cancel{-2}\sqrt{3}}{\cancel{-2}}$

$\tan(\theta) = \sqrt{3}$

$\tan(\theta) = \sqrt{3}$

$\theta = 60°$? (from basic trig function values).

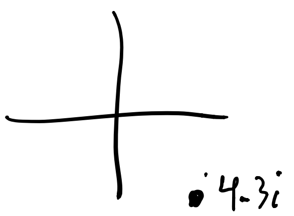
Quadrant I,

So $\theta = 60° + 180°$

$\theta = 240°$ Quadrant III

$-2-2\sqrt{3}i = \boxed{4(\cos 240° + i\sin 240°)}$

c) $4-3i$



Quadrant IV

modulus $r = \sqrt{4^2 + (-3)^2}$

$= \sqrt{16+9} = \sqrt{25} = 5$

$\boxed{r=5}$

argument θ

$\tan(\theta) = \frac{-3}{4}$

$\tan^{-1}\left(\frac{-3}{4}\right) \approx -36.9°$

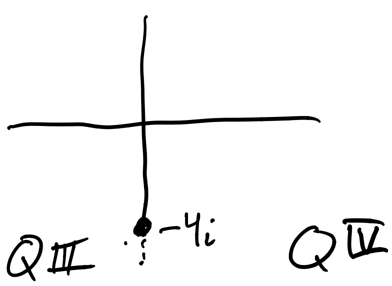
$\boxed{\theta = -36.9}$ ~~Quad~~ Quadrant IV

$4-3i = \boxed{5(\cos(-36.9°) + i\sin(-36.9°))}$

d) $-4i$

$= 0 - 4i$

a+bi?



modulus $r = \sqrt{0^2 + (-4)^2}$

$= \sqrt{16} = 4$

$r = 4$

$\tan(\theta) = \frac{-4}{0}$ = undefined

$\theta = 270°$ or $\theta = -90°$

$-4i = \boxed{4(\cos 270° + i\sin 270°)}$

Example 21.8
converting from
Polar Form
to
Standard form

Example 21.8. Convert the number from polar form into the standard form $a+bi$

a) $3 \cdot (\cos(117°) + i\sin(117°))$

b) $4 \cdot \left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$

$a+bi$

a) $3 \cdot (\cos(117°) + i\sin(117°))$

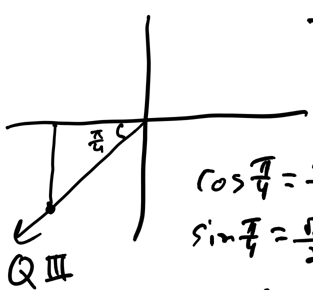
$= 3 \cdot (-.454 + i(.891))$

$= \boxed{-1.362 + 2.673i}$

b) $4\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$

$= 4\left(-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right)$

$= \boxed{-2\sqrt{2} - 2\sqrt{2}i}$



$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

$\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

Multiplying and dividing in Polar Form

Multiply in polar form:
multiply moduli (r's)
add arguments (θ's)

Divide in polar form
divide the moduli (r)
subtract the arguments (θ)

Proposition 21.9. Let $r_1\left(\cos\left(\theta_1\right)+i\sin\left(\theta_1\right)\right)$ and $r_2\left(\cos\left(\theta_2\right)+i\sin\left(\theta_2\right)\right)$ be two complex numbers in polar form. Then, the product and quotient of these are given by

$$r_1\left(\cos\left(\theta_1\right)+i\sin\left(\theta_1\right)\right)\cdot r_2\left(\cos\left(\theta_2\right)+i\sin\left(\theta_2\right)\right) = r_1r_2\cdot\left(\cos\left(\theta_1+\theta_2\right)+i\sin\left(\theta_1+\theta_2\right)\right)$$

$$\frac{r_1(\cos(\theta_1)+i\sin(\theta_1))}{r_2(\cos(\theta_2)+i\sin(\theta_2))} = \frac{r_1}{r_2}\cdot\left(\cos\left(\theta_1-\theta_2\right)+i\sin\left(\theta_1-\theta_2\right)\right)$$

Example 21.10. Multiply or divide the complex numbers, and write your answer in polar and standard form.

a) $5\left(\cos\left(11^\circ\right)+i\sin\left(11^\circ\right)\right)\cdot 8\left(\cos\left(34^\circ\right)+i\sin\left(34^\circ\right)\right)$

b) $3\left(\cos\left(\frac{5\pi}{8}\right)+i\sin\left(\frac{5\pi}{8}\right)\right)\cdot 12\left(\cos\left(\frac{7\pi}{8}\right)+i\sin\left(\frac{7\pi}{8}\right)\right)$

c) $\frac{32\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)}{8\left(\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)\right)}$

d) $\frac{4(\cos(203^\circ)+i\sin(203^\circ))}{6(\cos(74^\circ)+i\sin(74^\circ))}$

$r(\cos\theta + i\sin\theta)$

e) INTUITIVE BONUS: Without doing any calculation or conversion, describe where in the complex plane to find the number obtained by multiplying $(5+2i)(-1+6i)$.

a) $5\left(\cos(11^\circ)+i\sin(11^\circ)\right)\cdot 8\left(\cos(34^\circ)+i\sin(34^\circ)\right)$

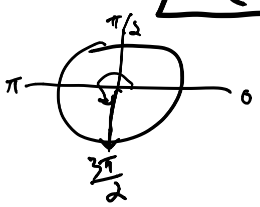
$= \boxed{40\left(\cos(45^\circ)+i\sin(45^\circ)\right)}$ standard form $= 40\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) = \boxed{20\sqrt{2}+20\sqrt{2}\,i}$

b) $3\left(\cos\frac{5\pi}{8}+i\sin\frac{5\pi}{8}\right)\cdot 12\left(\cos\frac{7\pi}{8}+i\sin\frac{7\pi}{8}\right)$

$r = 3\cdot 12 = 36$

$\theta = \frac{5\pi}{8}+\frac{7\pi}{8} = \frac{\cancel{12}^{3}\pi}{\cancel{8}_{2}} = \frac{3\pi}{2}$

$= \boxed{36\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)}$, standard form $36(0+i(-1)) = \boxed{-36i}$

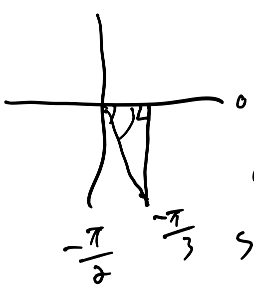


c) c) $\frac{32\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)}{8\left(\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)\right)} = \boxed{4\left(\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right)}$ standard form: $4\left(\frac{1}{2}+i\left(-\frac{\sqrt{3}}{2}\right)\right)$

$= \boxed{2-2\sqrt{3}\,i}$

$r = \frac{32}{8} = 4$

$\theta = \frac{\pi}{4}-\frac{7\pi}{12} = \frac{3\pi}{12}-\frac{7\pi}{12} = \frac{-4\pi}{12} = -\frac{\pi}{3}$



$\cos\frac{\pi}{3} = \frac{1}{2}$ $\quad \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ $\quad$ Q III

d) d) $\frac{4(\cos(203^\circ)+i\sin(203^\circ))}{6(\cos(74^\circ)+i\sin(74^\circ))} = \frac{2}{3}\left(\cos 129^\circ - i\sin 129^\circ\right)$

d)

$r = \frac{4}{6} = \frac{2}{3}$

$\theta = 203° - 74°$

$\theta = 129°$

~~cos 129° =~~

$\cos 129° = -.629$

$\sin 129° = .777$

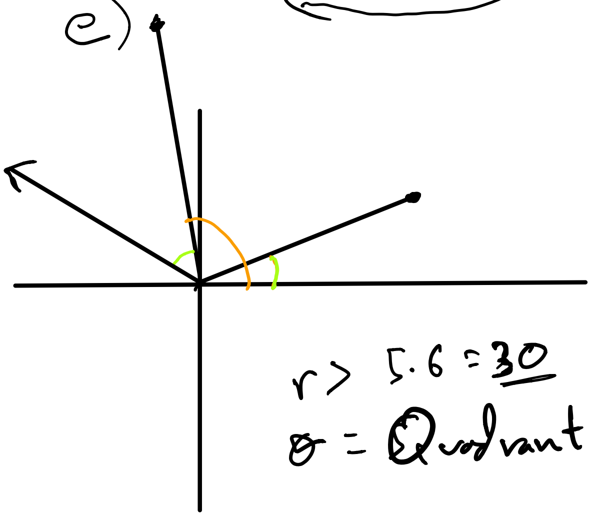
standard form:

$\frac{2}{3}(-.629 + i(.777))$

$= \boxed{-0.420 + 0.518i}$

$(5+2i)(-1+6i)$.

e)



$r > 5.6 = \underline{30}$

θ = Quadrant II