Trigonometric Equations of the form: $\tan(x) = C$

Example 20.1. Solve for x: $\tan(x) = \sqrt{3}$

one solution is $x = \boxed{\frac{\pi}{3}}$

also $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

$x = \frac{4\pi}{3} + \pi \cdot \frac{3}{3} = \frac{4\pi}{3} + \frac{3\pi}{3} = \boxed{\frac{7\pi}{3}}$

$x = \frac{7\pi}{3} + \pi = \frac{7\pi}{3} + \frac{3\pi}{3} = \boxed{\frac{10\pi}{3}}$

$\vdots$

$x = \frac{-2\pi}{3}$

$x = \frac{-2\pi}{3} - \pi = \frac{-2\pi}{3} - \frac{3\pi}{3} = \frac{-5\pi}{3}$

$x = \frac{-5\pi}{3} - \pi = \frac{-8\pi}{3}$

$\boxed{\text{solution: } x = \frac{\pi}{3} + n \cdot \pi}$ $n = 0, \pm 1, \pm 2, \ldots$

Observation 20.2. To solve $\tan(x) = c$, we first determine one solution $x = \tan^{-1}(c)$. Then the general solution is given by:

$$\boxed{x = \tan^{-1}(c) + n \cdot \pi} \text{ where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$n \in \mathbb{Z}$

n is an integer

$n \in \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

one solution goes here

$\boxed{\frac{\pi}{3} + n \cdot \pi, \quad n \text{ is an integer}}$

Trigonometric Equations of the form $\cos(x) = c$

Example 20.4. Solve for x: $\cos(x) = \frac{1}{2}$

x	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.

one solution $x = \frac{\pi}{3}$

$x = \frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$

$x = \frac{7\pi}{3} + 2\pi = \frac{13\pi}{3}$

$\boxed{x = \frac{\pi}{3} + 2\pi \cdot n}$ $n = 0, \pm 1, \pm 2, \ldots$

since $\cos(-x) = \cos(x)$ "$\cos(x)$ is even"

$x = -\frac{\pi}{3}$ is a solution

$x = -\frac{\pi}{3} + 2\pi = -\frac{\pi}{3} + \frac{6\pi}{3} = \frac{5\pi}{3}$

$x = \frac{5\pi}{3} + 2\pi = \ldots$

$x = -\frac{\pi}{3} - 2\pi = \frac{-7\pi}{3}$

$\boxed{x = -\frac{\pi}{3} + 2\pi \cdot n}$ $n = 0, \pm 1, \pm 2, \pm 3 \ldots$

$\boxed{\text{Solution: } x = \pm \frac{\pi}{3} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots}$

Observation 20.5. To solve $\cos(x) = c$, we first determine one solution $x = \cos^{-1}(c)$. Then the general solution is given by:

$$x = \pm\cos^{-1}(c) + 2n \cdot \pi \text{ where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Trigonometric Equations of the form $\sin(x) = c$

Example 20.7. Solve for x: $\sin(x) = \frac{\sqrt{2}}{2}$

$\boxed{x = \frac{\pi}{4}}$

$x = \frac{\pi}{4} + 2\pi = \frac{\pi}{4} + \frac{8\pi}{4} = \frac{9\pi}{4}$

$x = \frac{\pi}{4} + 4\pi = \frac{\pi}{4} + \frac{16\pi}{4} = \frac{17\pi}{4}$ $\quad \boxed{x = \frac{\pi}{4} + 2\pi n}$ $\quad n = 0, \pm 1, \pm 2, \dots$

recall:

$\sin(x) = \sin(\pi - x)$

$\boxed{\pi - \frac{\pi}{4}} = \frac{4\pi}{4} - \frac{\pi}{4} = \boxed{\frac{3\pi}{4}}$ *

$x = \frac{3\pi}{4} + 2\pi = \frac{3\pi}{4} + \frac{8\pi}{4} = \frac{11\pi}{4}$ * $\dots$

$\boxed{x = \frac{3\pi}{4} + 2\pi n}$ $\quad n = 0, \pm 1, \pm 2, \dots$

$\boxed{\underline{\text{Solution}} \quad x = (-1)^n \cdot \frac{\pi}{4} + n \cdot \pi}$ $\quad n = 0, \pm 1, \pm 2, \dots$

try plugging in $\underline{n}$

$n = 0, \quad x = (-1)^0 \cdot \frac{\pi}{4} + \cancel{0 \cdot \pi} = 1 \cdot \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$

$n = 2, \quad x = (-1)^2 \cdot \frac{\pi}{4} + 2 \cdot \pi$
$= 1 \cdot \frac{\pi}{4} + 2\pi = \frac{\pi}{4} + \frac{8\pi}{4} = \boxed{\frac{9\pi}{4}}$

$n = 4, \quad x = (-1)^4 \cdot \frac{\pi}{4} + 4 \cdot \pi$
$= \frac{\pi}{4} + 4\pi = \frac{\pi}{4} + \frac{16\pi}{4} = \boxed{\frac{17\pi}{4}}$

$n = 1, \quad x = (-1)^1 \cdot \frac{\pi}{4} + 1 \cdot \pi$
$= -\frac{\pi}{4} + \pi = -\frac{\pi}{4} + \frac{4\pi}{4} = \boxed{\frac{3\pi}{4}}$

$n = 3, \quad x = (-1)^3 \cdot \frac{\pi}{4} + 3 \cdot \pi$
$= -\frac{\pi}{4} + 3\pi = -\frac{\pi}{4} + \frac{12\pi}{4} = \boxed{\frac{11\pi}{4}}$

$n =$

Observation 20.8. To solve $\sin(x) = c$, we first determine one solution $x = \sin^{-1}(c)$. Then the general solution is given by:

$$x = (-1)^n \cdot \sin^{-1}(c) + n \cdot \pi \text{ where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Equations with trig functions (linear)

Example 20.11. Solve for x

a) $2\sin(x) - 1 = 0$

b) $\sec(x) = -\sqrt{2}$

c) $7\cot(x) + 3 = 0$

Summary: We summarize the different formulas used to solve the basic trigonometric equations in the following table.

Solve: $\sin(x) = c$	Solve: $\cos(x) = c$	Solve: $\tan(x) = c$
First, find one solution, that is: $\sin^{-1}(c)$. Use: $\sin^{-1}(-c) = -\sin^{-1}(c)$	First, find one solution, that is: $\cos^{-1}(c)$. Use: $\cos^{-1}(-c) = \pi - \cos^{-1}(c)$	First, find one solution, that is: $\tan^{-1}(c)$. Use: $\tan^{-1}(-c) = -\tan^{-1}(c)$
The general solution is: $x = (-1)^n \sin^{-1}(c) + n\pi$	The general solution is: $x = \pm\cos^{-1}(c) + 2n\pi$	The general solution is: $x = \tan^{-1}(c) + n\pi$
where $n = 0, \pm 1, \pm 2, \ldots$	where $n = 0, \pm 1, \pm 2, \ldots$	where $n = 0, \pm 1, \pm 2, \ldots$

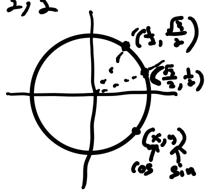
a) $2\sin(x) - 1 = 0$

$\quad +1 \quad +1$

Recall: $\sin(x) = c$.

$$\frac{\cancel{2}\sin(x)}{2} = \frac{1}{2}$$

$\frac{1}{2}, \frac{\sqrt{3}}{2}$



$\sin(x) = \frac{1}{2}$

$x = \frac{\pi}{6}$

$$\boxed{\text{Solution } x = (-1)^n \cdot \frac{\pi}{6} + \pi \cdot n} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

b) $\sec(x) = -\sqrt{2}$

$\sin(x) = c$

$\cos(x) = c$

$\tan(x) = c$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\cancel{\cos(x)} \cdot \frac{1}{\cancel{\cos(x)}} = -\sqrt{2} \cdot \cos(x)$$

$$\frac{1}{-\sqrt{2}} = \frac{-\cancel{\sqrt{2}} \cdot \cos(x)}{-\cancel{\sqrt{2}}}$$

$$\frac{1}{-\sqrt{2}} = \cos(x)$$

$$\cos(x) = -\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$\cos(x) = -\frac{\sqrt{2}}{2}$$

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \pi - \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$= \pi - \frac{\pi}{4} = \boxed{\frac{3\pi}{4}}$$

$$\boxed{\text{Solution: } x = \pm\frac{3\pi}{4} + 2\pi \cdot n \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots}$$

c) $7\cot(x) + 3 = 0$

$\sin(x) = c$

$\cos(x) = c$

$$\cot(x) = \frac{1}{\tan(x)}$$

$$\boxed{\tan(x) = c}$$

$$7 \cdot \left(\frac{1}{\tan(x)}\right) + 3 = 0$$

$$-3 \quad -3$$

$$\frac{\cancel{7} \cdot \frac{1}{\tan(x)}}{\cancel{7}} = \frac{-3}{7}$$

$$\frac{1}{\tan(x)} = -\frac{3}{7}$$

$$\frac{\tan(x)}{1} = -\frac{7}{3}$$

$$\tan(x) = -7/3$$

$$x = \tan^{-1}(-7/3)$$

$$\boxed{x \approx -1.16590}$$

$$\boxed{\text{solution} \quad x = -1.16590 + n\cdot\pi \quad , n = 0, \pm 1, \pm 2, \dots}$$

extra c) → give me one more solution:

pick $n = 3$, $x = -1.16590 + 3\pi$

$$\boxed{x \approx 8.25888}$$

Equations with trigonometric functions (quadratic)

Example 20.12. Solve for x.
a) $\tan^2(x) + 2\tan(x) + 1 = 0$ b) $2\cos^2(x) - 1 = 0$

a) $(\tan(x))^2 + 2\tan(x) + 1 = 0$

$\tan(x) = c$

let $u = \tan(x)$
substitute:

$u^2 + 2u + 1 = 0$

solve for u

$(u+1)(u+1) = 0$

$u+1=0$ $\quad$ $u+1=0$

$u=-1$ $\quad$ $u=-1$

$\boxed{\tan(x)=-1}$ $\quad$ $\tan(x)=-1$

one solution:

$\tan^{-1}(-1) = -\tan^{-1}(1)$

$\boxed{x = -\frac{\pi}{4}}$

solution: $\boxed{x = -\frac{\pi}{4} + n\cdot\pi \quad , n = 0, \pm1, \pm2, \ldots}$

Summary: We summarize the different formulas used to solve the basic trigonometric equations in the following table.

Solve: $\sin(x) = c$	Solve: $\cos(x) = c$	Solve: $\tan(x) = c$
First, find one solution, that is: $\sin^{-1}(c)$. Use: $\sin^{-1}(-c) = -\sin^{-1}(c)$	First, find one solution, that is: $\cos^{-1}(c)$. Use: $\cos^{-1}(-c) = \pi - \cos^{-1}(c)$	First, find one solution, that is: $\tan^{-1}(c)$. Use: $\tan^{-1}(-c) = -\tan^{-1}(c)$
The general solution is: $x = (-1)^n \sin^{-1}(c) + n\pi$	The general solution is: $x = \pm\cos^{-1}(c) + 2n\pi$	The general solution is: $x = \tan^{-1}(c) + n\pi$
where $n = 0, \pm1, \pm2, \ldots$	where $n = 0, \pm1, \pm2, \ldots$	where $n = 0, \pm1, \pm2, \ldots$

b) $2\cos^2(x) - 1 = 0$

$2(\cos(x))^2 - 1 = 0$

let $u = \cos(x)$
substitute:

$2u^2 - 1 = 0$

$+1 \quad +1$

$ax^2 + bx + c = 0$

$$\frac{2u^2}{2} = \frac{1}{2}$$

$$\sqrt{u^2} = \pm\sqrt{\frac{1}{2}}$$

$$u = \pm\sqrt{\frac{1}{2}}$$

$$\cos(x) = \pm\sqrt{\frac{1}{2}}$$

$$\cos(x) = \pm\frac{\sqrt{1}}{\sqrt{2}}$$

$$\cos(x) = \pm\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$\boxed{\cos(x) = \pm\frac{\sqrt{2}}{2}}$$

$$\cos(x) = \frac{\sqrt{2}}{2}$$

$$\underline{\cos(x) = c}$$

$$x = \frac{\pi}{4}$$

$$\cos(x) = -\frac{\sqrt{2}}{2}$$

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \pi - \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$= \pi - \frac{\pi}{4} = \frac{4\pi}{4} - \frac{\pi}{4} = \boxed{\frac{3\pi}{4}}$$

$$\boxed{x = \pm\frac{\pi}{4} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots}$$

$$\boxed{x = \pm\frac{3\pi}{4} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots}$$