Example

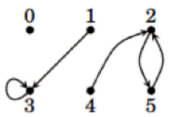
Let $A = \{1,2,3,4\}$, and consider the set
$R = \{(1,1),(2,1),(2,2),(3,3),(3,2),(3,1),(4,4),(4,3),(4,2),(4,1)\} \subseteq A \times A$.
1. True or false: a. 1R1 b. 2R1 c. 1R2 d. 4R4 e. 2R4
2. What does R mean? (*What familiar relation does R represent?)*

Example

Let $A = \{1,2,3,4\}$, and consider the set
$S = \{(1,1),(1,3),(3,1),(3,3),(2,2),(2,4),(4,2),(4,4)\} \subseteq A \times A$.
What does S mean?

Example

Here is a picture of a relation U on a set B.



Find the sets B and U.

Example

Consider the set $R = \{(x,x) : x \in \mathbb{R}\}$. What does R represent?

Example

Consider the set $A = \mathbb{Z}$, the integers. For each of the following relations, determine if it is reflexive, symmetric, transitive, antisymmetric or irreflexive
a. $<$ b. $\leq$ c. $=$ d. $\neq$

Example

Let $A = \{b,c,d,e\}$ and $R = \{(b,b), (b,c), (c,b), (c,c), (d,d), (b,d), (d,b), (c,d), (d,c)\}$
Determine whether R is reflexive, symmetric, transitive, antisymmetric or irreflexive.

Day 27

Sec 11.2, 11.3

- equivalence relation - equivalence classes	- partition

Definitions & Theorems

- A relation R on a set A is an **equivalence relation** if it is reflexive, symmetric and transitive.
- Suppose R is an equivalence relation on a set A. Given an element $a \in A$, the **equivalence class containing *a*** is the subset $\{x \in A : xRa\}$ of A consisting of all elements that are related to a. This is denoted $[a]$, so the equivalence class containing a is $[a] = \{x \in A : xRa\}$
  NOTE: beware! a is an element, but [a] is a SET - a collection of elements of A.
- A **partition** of a set A is a set of non-empty subsets of A such that the union of all the subsets equals A, and the intersection of two different subsets is $\varnothing$.
  *Basically, a **partition** is a division of A into subsets.*
- Theorem. Suppose R is an equivalence relation on a set A. Then the set $\{[a] : a \in A\}$ of equivalence classes of R forms a partition of A.

Example 1

Your group will be assigned one of the relations below on the set $A = \{-1,1,2,3,4\}$.
a. Draw a diagram of the relation.
b. Determine whether the relation is an equivalence relation (be prepared to explain).
c. State in a few words what the relation represents.

R_1 = {(-1,-1), (1,1), (2,2), (3,3), (4,4), (-1,1),(1,-1),(-1,3),(3,-1),(1,3),(3,1),(2,4),(4,2)}
R_2 = {(-1,1),(-1,2),(-1,3),(-1,4),(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}
R_3 = {(-1,-1), (1,1), (1,2),(2,1),(2,2),(2,3),(3,2),
(3,3),(1,3),(3,1),(4,4),(1,4),(4,1),(2,4),(4,2),(3,4),(4,3)}
R_4 = {(2,2),(3,3),(-1,-1),(4,4),(1,1)}
R_5 = (4,4),(4,3),(4,2),(4,1),(4,-1),(3,3),(3,2),
(3,1),(3,-1),(2,2),(2,1),(2,-1),(1,1),(1,-1),(-1,-1)}

Example 2

Example 2. a. For equivalence relation R_1, what is [1]? What is [4]? What is [-1]?
b. For each equivalence relation in Example 1, list the equivalence classes.

Example 3

RECALL: Given integers a and b and an $n \in \mathbb{N}$, then $a \equiv b \pmod{n}$ if $n|(a-b)$.
We say "a and b are congruent modulo n."

Consider the relation R on the integers given by aRb if and only if $a \equiv b \pmod{4}$.

1. True or false: a. 4R8 b. 4R9 c.5R7 d. 5R9 e. 3R11 f. 2R18
2. How can you tell whether two numbers are congruent mod 4?
3. Is R reflexive? Prove your answer.
4. Is R symmetric? Prove your answer.
5. Is R transitive? Prove your answer.
6. Is R an equivalence relation?
7. What is the equivalence class [4] = ?
8. How many equivalence classes are there? List them.

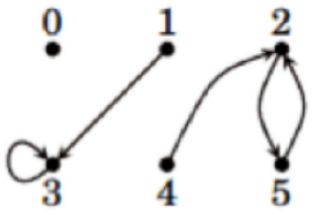
Example 4

a. How many equivalence classes will there be for the equivalence relation $a \equiv b \pmod{5}$? List the classes.

b. For a given natural number n, how many different equivalence classes will there be for the equivalence relation $a \equiv b \pmod{n}$?

Example

Here is a picture of a relation U on a set B.



Find the sets B and U.

Is it irreflexive? No
$(3,3) \in U$

Is it reflexive? No $(1,1) \notin U$

Is it antisymmetric?
$(2,5) \in U$
$(5,2) \in U$
BUT $2 \neq 5$
Not antisymmetric.

underlying set $B = \{0,1,2,3,4,5\}$

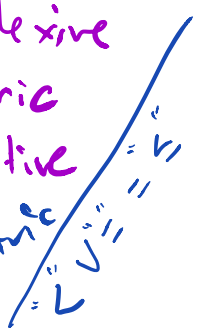
relation $U = \{(3,3), (1,3), (4,2), (2,5), (5,2)\}$

Is it transitive? No ~ $(4,2) \in U$ and $(2,5) \in U$
BUT $(4,5) \notin U$.
NOT Transitive
$(2,5)$ and $(5,2) \in U$
but $(2,2) \notin U$
also $(5,5) \notin U$

Is it symmetric?
no $(1,3) \in U$
but $(3,1) \notin U$
$(4,2) \in U$
but $(2,4) \notin U$.

Some common relations. Underlying set is $\mathbb{Z}$

	$<$	$\leq$	$=$	$\neq$
ex:	$(1,3)$	$(1,3)$	$(1,1)$, $(3,3)$	$(1,3)$, $(3,1)$
	$(1,1) \notin <$	$(1,1)$		
	not reflexive	reflexive	reflexive	not reflexive
	not symmetric	not symmetric	symmetric	symmetric
	transitive	transitive	transitive	not transitive
	antisymmetric	antisymmetric	antisymmetric	not antisymmetric



Properties of relations

Defn a relation R on a set A is ... reflexive if $\forall x \in A,\ (x,x) \in R$

Defn "R on A is symmetric, $\forall x,y \in A\ (x,y) \in R \rightarrow (y,x) \in R$

Defn "R on A is transitive $\forall x,y,z \in A\ ((x,y) \in R \wedge (y,z) \in R) \rightarrow (x,z) \in R$

$(1,1)$ $(1,0)$ $(1,1)$

Defn "R on A is antisymmetric if $\forall x,y \in A$

$((x,y) \in R \wedge (y,x) \in R) \rightarrow x = y$

$\rightarrow$ $(1,3)$ and ~~$(3,1)$~~, but if 3

~~$(5,5)$~~ and $(3,5)$

$(7,10)$ and ~~$(10,7)$~~

No
counter example

Defn "R on A is irreflexive if
$\forall x \in A,\ (x,x) \notin R$

Empty relation $A = \{\}$
$R = \{\}$

Defn A relation R on a set A is an equivalence relation if R is reflexive, symmetric, and transitive.