Example

Let $A = \{1,2,3,4\}$, and consider the set

$R = \{(1,1),(2,1),(2,2),(3,3),(3,2),(3,1),(4,4),(4,3),(4,2),(4,1)\} \subseteq A \times A$.

1. True or false: a. 1R1 b. 2R1 c. 1R2 d. 4R4 e. 2R4
2. What does R mean? (*What familiar relation does R represent?)*

Example

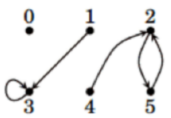
Let $A = \{1,2,3,4\}$, and consider the set

$S = \{(1,1),(1,3),(3,1),(3,3),(2,2),(2,4),(4,2),(4,4)\} \subseteq A \times A$.

What does S mean?

Example

Here is a picture of a relation U on a set B.



Find the sets B and U.

Example

Consider the set $R = \{(x,x) \ : \ x \in \mathbb{R}\}$. What does R represent?

Example

Consider the set $A = \mathbb{Z}$, the integers. For each of the following relations, determine if it is reflexive, symmetric, transitive, antisymmetric or irreflexive

a. $<$ b. $\leq$ c. $=$ d. $\neq$

Example

Let $A = \{b,c,d,e\}$ and $R = \{(b,b),\ (b,c),\ (c,b),\ (c,c),\ (d,d),\ (b,d),\ (d,b),\ (c,d),\ (d,c)\}$

Determine whether R is reflexive, symmetric, transitive, antisymmetric or irreflexive.

Day 27

Sec 11.2, 11.3

- equivalence relation - equivalence classes	- partition

Definitions & Theorems

- A relation R on a set A is an **equivalence relation** if it is reflexive, symmetric and transitive.
- Suppose R is an equivalence relation on a set A. Given an element $a \in A$, the **equivalence class containing *a*** is the subset $\{x \in A : xRa\}$ of A consisting of all elements that are related to a. This is denoted $[a]$, so the equivalence class containing a is $[a] = \{x \in A : xRa\}$
  NOTE: beware! a is an element, but [a] is a SET - a collection of elements of A.
- A **partition** of a set A is a set of non-empty subsets of A such that the union of all the subsets equals A, and the intersection of two different subsets is $\varnothing$.
  *Basically, a **partition** is a division of A into subsets.*
- Theorem. Suppose R is an equivalence relation on a set A. Then the set $\{[a] : a \in A\}$ of equivalence classes of R forms a partition of A.

Example 1

Your group will be assigned one of the relations below on the set $A = \{-1,1,2,3,4\}$.

a. Draw a diagram of the relation.

b. Determine whether the relation is an equivalence relation (be prepared to explain).

c. State in a few words what the relation represents.

$R_1 = \{(-1,-1), (1,1), (2,2), (3,3), (4,4), (-1,1),(1,-1),(-1,3),(3,-1),(1,3),(3,1),(2,4),(4,2)\}$

$R_2 = \{(-1,1),(-1,2),(-1,3),(-1,4),(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

$R_3 = \{(-1,-1), (1,1), (1,2),(2,1),(2,2),(2,3),(3,2),$
$(3,3),(1,3),(3,1),(4,4),(1,4),(4,1),(2,4),(4,2),(3,4),(4,3)\}$

$R_4 = \{(2,2),(3,3),(-1,-1),(4,4),(1,1)\}$

$R_5 = (4,4),(4,3),(4,2),(4,1),(4,-1),(3,3),(3,2),$
$(3,1),(3,-1),(2,2),(2,1),(2,-1),(1,1),(1,-1),(-1,-1)\}$

Example 2

Example 2. a. For equivalence relation R_1, what is [1]? What is [4]? What is [-1]?

b. For each equivalence relation in Example 1, list the equivalence classes.

Example 3

RECALL: Given integers a and b and an $n \in \mathbb{N}$, then $a \equiv b\ (mod\ n)$ if $n|(a-b)$.
We say "a and b are congruent modulo n."

Consider the relation R on the integers given by aRb if and only if $a \equiv b\ (mod\ 4)$.

1. True or false: a. 4R8 b. 4R9 c.5R7 d. 5R9 e. 3R11 f. 2R18
2. How can you tell whether two numbers are congruent mod 4?
3. Is R reflexive? Prove your answer.
4. Is R symmetric? Prove your answer.
5. Is R transitive? Prove your answer.
6. Is R an equivalence relation?
7. What is the equivalence class [4] = ?
8. How many equivalence classes are there? List them.

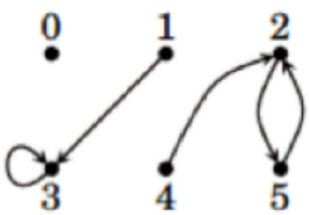
Example 4

a. How many equivalence classes will there be for the equivalence relation $a \equiv b\ (mod\ 5)$? List the classes.

b. For a given natural number n, how many different equivalence classes will there be for the equivalence relation $a \equiv b\ (mod\ n)$?

Example

Here is a picture of a relation U on a set B.



Find the sets B and U.

Is it irreflexive? No $(3,3) \in U$

Is it reflexive? No $(1,1) \notin U$

Is it antisymmetric?
$(2,5) \in U$
$(5,2) \in U$
But $2 \neq 5$
Not antisymmetric.

underlying set $B = \{0,1,2,3,4,5\}$

relation $U = \{(3,3), (1,3), (4,2), (2,5), (5,2)\}$

Is it symmetric?
no $(1,3) \in U$ but $(3,1) \notin U$
$(4,2) \in U$ but $(2,4) \notin U$.

Is it transitive? No ~ $(4,2) \in U$ and $(2,5) \in U$
But $(4,5) \notin U$.

Not Transitive

$(2,5)$ and $(5,2) \in U$
but $(2,2) \notin U$
also $(5,5) \notin U$

Some common relations. Underlying set is $\mathbb{Z}$

	$<$	$\leq$	$=$	$\neq$
ex:	$(1,3)$	$(1,3)$	$(1,1)$, $(3,3)$	$(1,3)$, $(3,1)$
	$(1,1) \notin <$	$(1,1)$		
	not reflexive	reflexive	reflexive	not reflexive
	not symmetric	not symmetric	symmetric	symmetric
	transitive	transitive	transitive	not transitive
	antisymmetric	antisymmetric	antisymmetric	not antisymmetric

Properties of relations

Defn a relation R on a set A is reflexive if $\forall x \in A, (x,x) \in R$

Defn " R on A is symmetric, $\forall x,y \in A \; (x,y) \in R \to (y,x) \in R$

Defn " R on A is transitive $\forall x,y,z \in A \; ((x,y) \in R \wedge (y,z) \in R) \to (x,z) \in R$

$(1,1)$ $(1,1)$ $(1,1)$

Defn " R on A is antisymmetric if $\forall x,y \in A$

$$((x,y) \in R \wedge (y,x) \in R) \to x=y$$

$\to$ $(1,3)$ and $\cancel{(3,1)}$, but $1 \neq 3$

$\cancel{(5,3)}$ and $(3,5)$

$(7,10)$ and $\cancel{(10,7)}$

A (

No
counter example

Defn "R on A" is irreflexive if
$\forall x \in A, (x,x) \notin R$

Empty relation $A = \{\}$
$R = \{\}$

Defn A relation R on a set A is an equivalence relation if R is reflexive, symmetric, and transitive.

$A = \{-1, 1, 2, 3, 4\}$

$R_1 = \{(-1,-1), (1,1), (2,2), (3,3), (4,4), (-1,1),(1,-1),(-1,3),(3,-1),(1,3),(3,1),(2,4),(4,2)\}$

Is R_1 an equivalence relation? yes – it is reflexive, symmetric, transitive.

① is it reflexive? Yes.

② is it symmetric? Yes.

③ is it transitive? Yes. Find all x, y, z with $(x,y) \in R_1$ and $(y,z) \in R_1$, and check if $(x,z) \in R_1$.

(x,y)	(y,z)	(x,z)
$(4,4) \in R$	$(4,2) \in R$	$(4,2) \in R$?
$(3,1) \in R$	$(1,-1)$	$(3,-1) \in R$ ✓
$(1,3)$	$(3,-1)$	$(1,-1) \in R$ ✓
$(2,4)$	$(4,2)$	$(2,2) \in R$ ✓
$\cancel{(1,2)} \in R$	$(2,4) \in R$?	$\cancel{(1,4) \in R}$

R on A

equivalence classes → $\{2,4\}$ ← everything here is connected

$\{3,1,-1\}$ ← everything here is connected.

IDEA an equivalence relation R divides A up into subsets called <u>equivalence classes</u>

$A = \{-1,1,2,3,4\}$

R_1

$[a]$ = the equivalence class containing a.

$a \in A$

$[-1] = \{-1, 1, 3\}$

$[2] = \{2, 4\}$

$[4] = \{2, 4\}$

$[2] = [4]$

Qus does $[-1] \neq [4]$?

$\{-1, 1, 3\}$ $\{2, 4\}$

Defn if $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$, then we say <u>a is congruent to b modulo n</u> or write <u>$a \equiv b \bmod n$</u>, if $n \mid (b-a)$

Ex $n=4$

Q: is $5 \equiv 3 \bmod 4$? No

$3-5=-2$

does $4 \mid -2$? No

is $5 \equiv 13 \bmod 4$? Yes

$13-5=8$, $4 \mid 8$ ✓

is $13 \equiv 5 \bmod 4$? Yes

$5-13=-8$, $4 \mid -8$

Ex: Consider relation R on $\mathbb{Z}$

given by: xRy if $x \equiv y \bmod 4$

$(x,y) \in R$ if $x \equiv y \bmod 4$

$(1,5) \in R$ $\quad 1 \equiv 5 \bmod 4$?

$5-1=4$

$4 \mid 4$ ✓

$(5,13) \in R$ $\rightarrow$ $5 \equiv 13 \bmod 4$

is $(6,2) \in R$? Yes, is $6 \equiv 2 \bmod 4$ ✓

is $(2,6) \in R$? yes $2-6=-4$

$4 \mid -4$ ✓

$2 \equiv 6 \mod 4$

$6-2=4$

$4 \mid 4$ ✓

Ques is R reflexive?

is R symmetric?

is R transitive?

is R an equivalence relation?

Last hw is optional (extra cred) due next Tues.

can you prove your answer?

Recall **Defn** relation on $\mathbb{Z}$

$x, y \in \mathbb{Z}$, $xRy \longleftrightarrow x \equiv y \bmod 4$

$(6,2) \in R$

other things related to 6

$(6, ?)$

$(6,6) \in R$ $\quad 6-6=0, \quad 4 \mid 0$

$(6,-36) \notin R$ $\quad 6-(-36)=42 \quad 4 \nmid 42$

$(6,-34) \in R$ $\quad 6-(-34)=40 \quad 4 \mid 40$

$(6,18) \in R$ $\quad 6-18=-12 \quad 4 \mid -12$

$(6,22) \in R$ $\quad 6-22=-16 \quad 4 \mid -16$

$(6,26) \in R$ $\quad 6-26=-20 \quad 4 \mid -20$

Is R reflexive?

Prop R (congruence mod 4) is reflexive.

translation: **Prop** $\forall x \in \mathbb{Z},\ xRx$

translation **Prop** $\forall x \in \mathbb{Z} \quad x \equiv x \bmod 4$

could say aRa is always true.

Prop $\forall x \in \mathbb{Z}$ $x \equiv$ ...

always true or $a \equiv a \bmod 4$ $4 \mid a-a$, $4 \mid 0$

Proof. Suppose $a \in \mathbb{Z}$

Notice: $4 \mid 0$ since $0 = 4 \cdot 0$ by defn of "$\mid$"

$4 \mid a - a$ by substitution $0 = a - a$.

Thus $a \equiv a \bmod 4$. ☑

thus R is reflexive ☑

symmetric

$\boxed{\text{Prop. } \forall x, y \in \mathbb{Z} \left(x \equiv y \bmod 4 \rightarrow y \equiv x \bmod 4 \right)}$

Proof Suppose $x, y \in \mathbb{Z}$ and $x \equiv y \bmod 4$

so $4 \mid y - x$ by defn of $\equiv \bmod 4$.

so $y - x = 4a$, some $a \in \mathbb{Z}$, by defn "$\mid$"

multiply through by -1 to obtain:

$-y + x = -4a$

$x - y = 4(-a)$ note: $-a \in \mathbb{Z}$ by clos.

$4 \mid x - y$ defn of "$\mid$"

divides symbol

Thus $y \equiv x \bmod 4$.

Thus R is symmetric ☑

transitivity

if xRy and yRz then xRz

Prop $\forall x, y, z \in \mathbb{Z} \left(x \equiv y \bmod 4 \wedge y \equiv z \bmod 4 \right) \rightarrow x \equiv z \bmod 4$.

Proof Suppose $x, y, z \in \mathbb{Z}$ and $x \equiv y \bmod 4$ and $y \equiv z \bmod 4$.

$4 \mid y-x$ and $4 \mid z-y$ by defn $\equiv \bmod 4$.

thus $y-x=4m$ $z-y=4n$, $n,m \in \mathbb{Z}$ by defn of "$\mid$"

so $y=4m+x$ (algebra), and substituting, we obtain

$$z-(4m+x)=4n$$

$$z-4m-x=4n$$

$$\underset{+4m}{} \qquad \underset{+4m}{}$$

$$z-x=4n+4m=4(n+m),$$ note $n+m \in \mathbb{Z}$ by closure

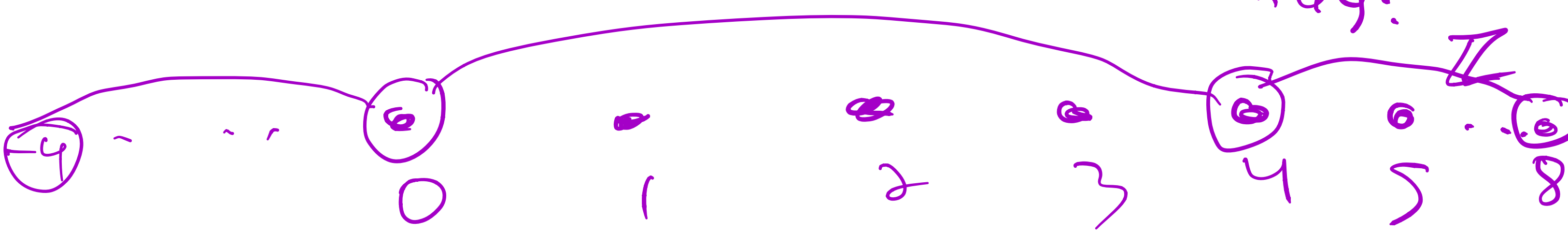
So $4 \mid z-x$ by defn of "$\mid$"

Thus $x \equiv z \bmod 4$ by defn $\equiv \bmod 4$.

Thus R is transitive. $\square$

Thus R is reflexive, symmetric and transitive, and so R is a equivalence relation.

Equivalence classes? How many? What are they?



$\{\ldots -8, -4, 0, 4, 8, 12, 16, \ldots\}$ ← one equivalence class

there are 4 equiv. classes

all numbers divisible by 4

$\boxed{\equiv \bmod n \text{ has } n \text{ equivalence classes}}$

$\{1, 5, 9, 13, -3, -7, \ldots\}$

all numbers with remainder 1 when divided by 4.

$\{\ldots, 2, 6, 10\}$

remainder 2

$\{-1, 3, 7, 11, \ldots\}$

remainder 3 when divided by 4,

Directly related to remainders and

Chinese remainder theorem/
Division algorithm