

Example

Let $A = \{1, 2, 3, 4\}$, and consider the set
 $R = \{(1, 1), (2, 1), (2, 2), (3, 3), (3, 2), (3, 1), (4, 4), (4, 3), (4, 2), (4, 1)\} \subseteq A \times A$.

1. True or false: a. 1R1 b. 2R1 c. 1R2 d. 4R4 e. 2R4
 2. What does R mean? (*What familiar relation does R represent?*)

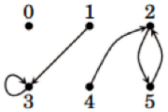
Example

Let $A = \{1, 2, 3, 4\}$, and consider the set
 $S = \{(1, 1), (1, 3), (3, 1), (3, 3), (2, 2), (2, 4), (4, 2), (4, 4)\} \subseteq A \times A$.

What does S mean?

Example

Here is a picture of a relation U on a set B.



Find the sets B and U.

Example

Consider the set $R = \{(x, x) : x \in \mathbb{R}\}$. What does R represent?

Example

Consider the set $A = \mathbb{Z}$, the integers. For each of the following relations, determine if it is reflexive, symmetric, transitive, antisymmetric or irreflexive

a. $<$ b. \leq c. $=$ d. \neq

Example

Let $A = \{b, c, d, e\}$ and $R = \{(b, b), (b, c), (c, b), (c, c), (d, d), (b, d), (d, b), (c, d), (d, c)\}$

Determine whether R is reflexive, symmetric, transitive, antisymmetric or irreflexive.

Day 27

Sec 11.2, 11.3

- | | |
|---|-------------|
| - equivalence relation
- equivalence classes | - partition |
|---|-------------|

Definitions & Theorems

- A relation R on a set A is an **equivalence relation** if it is reflexive, symmetric and transitive.
- Suppose R is an equivalence relation on a set A . Given an element $a \in A$, the **equivalence class containing a** is the subset $\{x \in A : xRa\}$ of A consisting of all elements that are related to a . This is denoted $[a]$, so the equivalence class containing a is $[a] = \{x \in A : xRa\}$
NOTE: beware! a is an element, but $[a]$ is a SET - a collection of elements of A .
- A **partition** of a set A is a set of non-empty subsets of A such that the union of all the subsets equals A , and the intersection of two different subsets is \emptyset .
Basically, a partition is a division of A into subsets.
- Theorem. Suppose R is an equivalence relation on a set A . Then the set $\{[a] : a \in A\}$ of equivalence classes of R forms a partition of A .

Example 1

Your group will be assigned one of the relations below on the set $A = \{-1, 1, 2, 3, 4\}$.

- a. Draw a diagram of the relation.
 b. Determine whether the relation is an equivalence relation (be prepared to explain).
 c. State in a few words what the relation represents.

$$R_1 = \{(-1, -1), (1, 1), (2, 2), (3, 3), (4, 4), (-1, 1), (1, -1), (-1, 3), (3, -1), (1, 3), (3, 1), (2, 4), (4, 2)\}$$

$$R_2 = \{(-1, -1), (-1, 2), (-1, 3), (-1, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

$$R_3 = \{(-1, -1), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2),$$

$$(3, 3), (1, 3), (3, 1), (4, 4), (1, 4), (4, 1), (2, 4), (4, 2), (3, 4), (4, 3)\}$$

$$R_4 = \{(2, 2), (3, 3), (-1, -1), (4, 4), (1, 1)\}$$

$$R_5 = (4, 4), (4, 3), (4, 2), (4, 1), (4, -1), (3, 3), (3, 2),$$

$$(3, 1), (3, -1), (2, 2), (2, 1), (2, -1), (1, 1), (1, -1), (-1, -1)\}$$

Example 2

- Example 2. a. For equivalence relation R_1 , what is $[1]$? What is $[4]$? What is $[-1]$?
 b. For each equivalence relation in Example 1, list the equivalence classes.

Example 3

RECALL: Given integers a and b and an $n \in \mathbb{N}$, then $a \equiv b \pmod{n}$ if $n|(a-b)$.
We say " a and b are congruent modulo n ."

Consider the relation R on the integers given by aRb if and only if $a \equiv b \pmod{4}$.

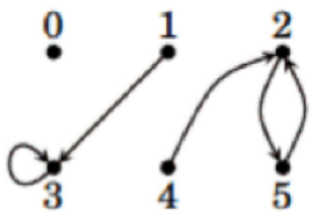
1. True or false: a. $4R8$ b. $4R9$ c. $5R7$ d. $5R9$ e. $3R11$ f. $2R18$
2. How can you tell whether two numbers are congruent mod 4?
3. Is R reflexive? Prove your answer.
4. Is R symmetric? Prove your answer.
5. Is R transitive? Prove your answer.
6. Is R an equivalence relation?
7. What is the equivalence class $[4]$?
8. How many equivalence classes are there? List them.

Example 4

- a. How many equivalence classes will there be for the equivalence relation $a \equiv b \pmod{5}$? List the classes.
- b. For a given natural number n , how many different equivalence classes will there be for the equivalence relation $a \equiv b \pmod{n}$?

Example

Here is a picture of a relation U on a set B .



Find the sets B and U .

Is it irreflexive? No
 $(3,3) \in U$

Is it reflexive? No
 $(1,1) \notin U$

Is it antisymmetric?
 $(2,5) \in U$
 $(5,2) \in U$
BUT $2 \neq 5$
Not antisymmetric.

underlying set

$B = \{0, 1, 2, 3, 4, 5\}$

relation

$U = \{(3,3), (1,3), (4,2), (2,5), (5,2)\}$

Is it transitive? No ~ $(4,2) \in U$ and $(2,5) \in U$
BUT $(4,5) \notin U$.

NOT Transitive

$(2,5)$ and $(5,2) \in U$
but $(2,2) \notin U$
also $(5,5) \notin U$

Is it symmetric?
no $(1,3) \in U$
but $(3,1) \notin U$

$(4,2) \in U$
but $(2,4) \notin U$.

Some common relations. Underlying set is \mathbb{Z}

	<u><</u>	<u>≤</u>	<u>=</u>	<u>≠</u>
<u>ex:</u>	(1,3)	(1,3)	(1,1), (3,3)	(1,3), (3,1)
	(1,1) ∉ <	(1,1)		
	not reflexive	reflexive	reflexive	not reflexive
	not symmetric	not symmetric	symmetric	symmetric
	transitive	transitive	transitive	not transitive
	antisymmetric	antisymmetric	antisymmetric	not antisymmetric

Properties of relations

Defn a relation R on set A is reflexive if $\forall x \in A, (x,x) \in R$

Defn " R on A is symmetric, $\forall x,y \in A (x,y) \in R \rightarrow (y,x) \in R$

Defn " R on A is transitive $\forall x,y,z \in A (x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R$

Defn " R on A is antisymmetric if $\forall x,y \in A$
 $((x,y) \in R \wedge (y,x) \in R) \rightarrow x=y$

$\rightarrow (1,3)$ and ~~(3,1)~~ but $\neq 3$

~~(5,3)~~ and (3,5)

(7,10) and ~~(10,7)~~

\wedge

No
counter example

Defn "R on A is irreflexive if
 $\forall x \in A, (x, x) \notin R$

Empty relation

$$A = \{\}$$

$$R = \{\}$$

Defn A relation R on a set A
is an equivalence relation
if R is reflexive, symmetric,
and transitive.

$$A = \{-1, 1, 2, 3, 4\}$$

$$R_1 = \{(-1, -1), (1, 1), (2, 2), (3, 3), (4, 4), (-1, 1), (1, -1), (-1, 3), (3, -1), (1, 3), (3, 1), (2, 4), (4, 2)\}$$

Is R_1 an equivalence relation? Yes - it is reflexive, symmetric, transitive.

- ① is it reflexive? Yes.
- ② is it symmetric? Yes.

③ is it transitive? Yes find all x, y, z

with $(x, y) \in R_1$
and $(y, z) \in R_1$
and check if $(x, z) \in R_1$.

(x, y)	(y, z)	(x, z)
$(4, 4) \in R_1$	$(4, 2) \in R_1$	$(4, 2) \in R_1$
$(3, 1) \in R_1$	$(1, -1) \in R_1$	$(3, -1) \in R_1$ ✓
$(1, 3) \in R_1$	$(3, -1) \in R_1$	$(1, -1) \in R_1$ ✓
$(2, 4) \in R_1$	$(4, 2) \in R_1$	$(2, 2) \in R_1$ ✓
$(1, 2) \in R_1$	$(2, 4) \in R_1$	$(1, 4) \in R_1$



$\{2, 4\}$ ← every thing here is connected

$\{3, 1, -1\}$ ← every thing here is connected.

equivalence classes

IDEA an equivalence relation R divides A up into subsets called equivalence classes

$$A = \{-1, 1, 2, 3, 4\}$$

R_1

$[a]$ = the equivalence class containing a .

$a \in A$

$$[-1] = \{-1, 1, 3\}$$

$$[2] = \{2, 4\}$$

$$[4] = \{2, 4\}$$

$$[2] = [4]$$

Qus does $[-1] \neq [4]$?

$$\{-1, 1, 3\} \quad \{2, 4\}$$

Defn if $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$, then
we say a is congruent to b modulo n
or write $a \equiv b \pmod{n}$, if
 $n \mid (b-a)$

Ex $n=4$

Q: is $5 \equiv 3 \pmod{4}$? No

$$3 - 5 = -2$$

does $4 \mid -2$? No

is $5 \equiv 13 \pmod{4}$? Yes

$$13 - 5 = 8, 4 \mid 8 \checkmark$$

is $13 \equiv 5 \pmod{4}$? Yes

$$5 - 13 = -8, 4 \mid -8$$

Ex:

consider relation R on \mathbb{Z}

given by: $x R y$ if $x \equiv y \pmod{4}$

$(x, y) \in R$ if $x \equiv y \pmod{4}$

$(1, 5) \in R$ $1 \equiv 5 \pmod{4}$?

$$5 - 1 = 4$$

$(5, 13) \in R$

$$4 \mid 4 \checkmark$$

$\rightarrow 5 \equiv 13 \pmod{4}$

is $(6, 2) \in R$? Yes, is $6 \equiv 2 \pmod{4}$ \checkmark

is $(2,6) \in R$? Yes $2-6 = -4$
 $4|-4$ ✓

$$2 \equiv 6 \pmod{4}$$

$$6-2 = 4$$

$$4|4 \checkmark$$

Ques is R reflexive?

is R symmetric?

is R transitive?

is R an equivalence relation?

Last homework is optional
(extra cred) due next
Tues.

→ can you prove your
answer?