Day 26

Sec 11.0, 11.1

- relations - reflexive - symmetric	- transitive - antisymmetric - irreflexive

Definitions & Theorems

- A **relation** R on a set A is a subset $R \subseteq A \times A$. We often abbreviate the statement $(x,y) \in R$ as xRy. The statement $(x,y) \notin R$ is abbreviated $\sim xRy$ or $x \not{R}\, y$.
- Suppose R is a relation on a set A.
  1. Relation R is **reflexive** if xRx for every $x \in A$: $\forall x \in A,\ xRx$
  2. Relation R is **symmetric** if xRy implies yRx for all $x, y \in A$: $\forall x, y \in A,\ xRy \Longrightarrow yRx$
  3. Relation R is **transitive** if whenever xRy and yRz, then also xRz: $\forall x, y, z \in A,\ ((xRy \wedge yRz) \Longrightarrow xRz)$
  4. Relation R is **antisymmetric** if for $x, y \in A$, xRy and yRx implies $x = y$: $\forall x, y \in A$, $(xRy \wedge\ yRx) \Longrightarrow x = y$
  5. Relation R is **irreflexive** if $\sim xRx$ for all $x \in A$: $\forall x \in A,\ \sim xRx$

Example

Consider the set $A = \{1,2,3,4,5\}$, and the relation '<' (less than). Make a complete list of correct comparisons of members of A according to '<'. *(for example:* $1 < 2,\ 2 < 5,\ 3 < 4,$ *etc.).*

Example

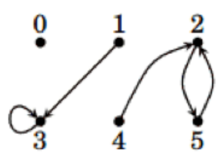
Let $A = \{1,2,3,4\}$, and consider the set
$R = \{(1,1),(2,1),(2,2),(3,3),(3,2),(3,1),(4,4),(4,3),(4,2),(4,1)\} \subseteq A \times A$.
1. True or false: a. 1R1 b. 2R1 c. 1R2 d. 4R4 e. 2R4
2. What does R mean? *(What familiar relation does R represent?)*

Example

Let $A = \{1,2,3,4\}$, and consider the set
$S = \{(1,1),(1,3),(3,1),(3,3),(2,2),(2,4),(4,2),(4,4)\} \subseteq A \times A$.
What does S mean?

Example

Here is a picture of a relation U on a set B.



Find the sets B and U.

Example

Consider the set $R = \{(x,x)\ :\ x \in \mathbb{R}\}$. What does R represent?

Example

Consider the set $A = \mathbb{Z}$, the integers. For each of the following relations, determine if it is reflexive, symmetric, transitive, antisymmetric or irreflexive

a. $<$ b. $\leq$ c. $=$ d. $\neq$

Example

Let $A = \{b,c,d,e\}$ and $R = \{(b,b),\ (b,c),\ (c,b),\ (c,c),\ (d,d),\ (b,d),\ (d,b),\ (c,d),\ (d,c)\}$
Determine whether R is reflexive, symmetric, transitive, antisymmetric or irreflexive.

Relation: describes a relationship or comparison between objects. Can be true or false.

Ex. $5 < 7$ T

$14 \geq 2$ T

$3 < 1$ F

$A \subseteq B$

$3 \mid 12$

$7 = 7$ T

$3 = 5$ F

Ex: let $A = \{1, 2, 3, 4, 5\}$

consider the relation "$<$" on A.

means that we are using $<$ to compare only elements of the set A.

$|A| < 10$ X

$|A| \notin A$

$10 \notin A$

$3 < pi$ X

$pi \notin A$

true statement

These 10 facts capture everything about $<$ on the set A.

$1<5$, $1<2$, $2<4$,

$1<3$, $1<4$, $2<3$,

$3<4$, $2<5$, $3<5$

$4<5$

let $R = \{(1,5), (1,2), (2,4), (1,3), (1,4), (2,3), (3,4), (2,5), (3,5), (4,5)\}$

we can use R to answer questions about $<$ on A.

ex: is $2<5$? look for $(2,5)\in R$. $2<5$ since $(2,5)\in R$, $2R5$

is $5<3$? look for $(5,3)$ in R. $5 \not< 3$ since $(5,3)\notin R$, $5 \not R 3$

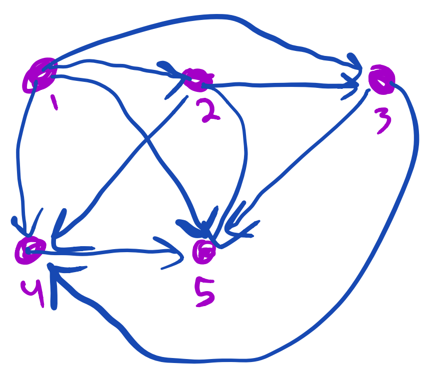
$\boxed{\text{IDEA: with the set } R\text{, we know everything about the relation } < \text{ on the set } A=\{1,2,3,4,5\}.}$

$\boxed{\text{Defn. A relation } R \text{ on a set } A \text{ is a subset } R \subseteq A\times A.}$

let $R=\{(1,5),(1,2),(2,4),(1,3),(1,4),(2,3),(3,4),(2,5),(3,5),(4,5)\}$

on

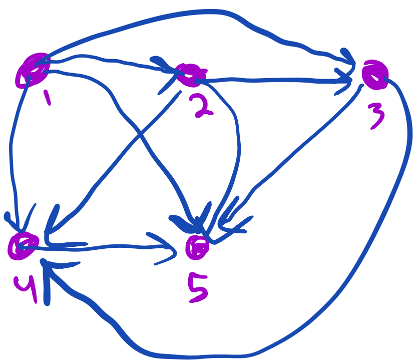
$A=\{1,2,3,4,5\}$



Picture of a relation as a directed graph

- one point for each element of A
- one arrow (directed edge) from x to y for each $(x,y) \in R$.

picture of $<$ on $\{1,2,3,4,5\}$,



looking at graph,

is $1<5$? Y, there is an arrow from 1 to 5.

is $4<2$? N, no arrow from 4 to 2.

next properties of relations

ex: reflexive

a relation R on a set A is reflexive if

$$\forall x \in A, \ (x,x) \in R$$

Q: is $<$ on $\{1,2,3,4,5\}$ reflexive? why or why not?

If it were reflexive,

$$\left.\begin{array}{l} (1,1) \notin R \\ (2,2) \notin R \\ (3,3) \notin R \end{array}\right\}$$

these are true, so $<$ is not reflexive.