Day 25

Chapter 10: Strong Induction

- recursion	- strong induction

Definitions & Theorems

- **Recursion** is a method for defining a sequence of numbers by giving one or more initial values (base cases), together with a rule for generating new members of the sequence from past members (the inductive case)
- **Strong induction** is a method for proving that a statement *P(n)* is true for all natural numbers *n*, by proving two associated statements:
  1. *P(1)* and 2. for any *k*, $P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(k) \Longrightarrow P(k+1)$
- NOTE: recursion and induction go hand-in-hand

Example. Consider the sequence of numbers $B_1,\ B_2,\ B_3,\ B_4,\ \ldots$ defined by the following three rules:
a. $B_1 = 4$ b. $B_2 = 12$ c. $for\ n \geq 3,\ B_n = B_{n-1} - 2B_{n-2}$

Proposition. For all n, $4|B_n$.

Outline for Proof by Strong Induction

Proposition. $\forall n \in \mathbb{N},\ P(n)$
Proof.
Base step. Prove $P(1)$ *(or the first several P(n), as necessary)*
Inductive step. Prove that for any given natural number *k*,
$(P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(k)) \Longrightarrow P(k+1)$.
It follows by mathematical induction that for every *n*, *P(n)* is true.

Theorem NT 5.1: Every natural number $n > 1$ is either prime or divisible by a prime.

Theorem NT 5.2: Suppose *p* is prime and $a_1, a_2, a_3, \ldots, a_n$ are *n* integers, where $n \geq 2$. If $p|a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$, then $p|a_i$ for at least one of the a_i ($1 \leq i \leq n$).

Theorem NT 5.3: If n is an integer greater than 1 then n can be written as a product of primes
(HINT: Prove using strong induction. Consider two cases, when k+1 is prime, and when it is composite)

Recursion

Example $B_1, B_2, B_3, B_4, \ldots$

$$\boxed{\begin{aligned} &B_1 = 4 \\ &B_2 = 12 \\ &\text{for } n \geq 3,\ B_n = B_{n-1} - 2B_{n-2} \end{aligned}}$$

$\leftarrow$ recursive definition

$$B_3 = B_2 - 2B_1 = 12 - 2(4) = 4$$

$$B_4 = B_3 - 2B_2 = 4 - 2(12) = -20$$

$$B_5 = B_4 - 2B_3 = \qquad = -28$$

$$B_6 = \qquad = 12$$

$$B_7 = \qquad = 68$$

$$B_8 = B_7 - 2B_6 = 68 - 2(12) = 44$$

$\vdots$

Prop $\forall n \in \mathbb{N}$, $4 \mid B_n$. $\quad P(n)$

Proof (induction) base case $n=1$, $B_1 = 4$ and so $4 \mid B_1$.

base case $n=2$, $B_2 = 12$ ...

Base case $n=2$, $B_2=12$, so $4 \mid B_2$

inductive step suppose $k \in \mathbb{N}$, $k \geq 2$

Suppose $4 \mid B_k$, so $B_k = 4a$, $a \in \mathbb{Z}$ by defn of divides

Since $B_{k+1} = B_k - 2B_{k-1}$

then $B_{k+1} = 4 \cdot a - 2B_{k-1}$

PROBLEM: we need to know $4 \mid B_{k-1}$, that is, $P(k-1)$

Pause

$$B_{k+1} = B_k - 2B_{k-1}$$

$$B_{k+1} = B_{(k+1)-1} - 2B_{(k+1)-2}$$

Thus $4 \mid B_{k+1}$

B_n, $n = k+1$

Outline for Strong induction

Prop. $\forall n \in \mathbb{N}$, $P(n)$

Proof (strong induction)

Base case(s): $n=1$, $n=2$ ... prove $P(1)$, $P(2)$, etc... (# of base cases depend on Proposition)

Inductive case ...

Inductive case: Suppose $k \in \mathbb{N}$, $k \geq 2$

Suppose the proposition holds for all natural numbers from 1 to k

Suppose $P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(k)$

$\vdots$

Thus $P(k+1)$ $\square$

$B_1 = 4$

$B_2 = 12$

for $n \geq 3$, $B_n = B_{n-1} - 2B_{n-2}$

Prop $\forall n \in \mathbb{N}$, $4 \mid B_n$

Proof (strong induction)

Base step $n=1$, $B_1 = 4$ (given),

so $4 \mid B_1$

also $n=2$, $B_2 = 12$ (given)

so $4 \mid B_2$

Inductive step Suppose $k \in \mathbb{N}$, $k \geq 2$

inductive hypothesis or inductive assumption

Suppose the proposition is true for all numbers from 1 to k.

Since $B_{k+1} = B_k - 2(B_{k-1})$, by defn of B_n's

By inductive hypothesis, $4 | B_k$ and $4 | B_{k-1}$.

Thus $B_k = 4a$, $B_{k-1} = 4b$ for $a, b \in \mathbb{Z}$ by defn of divides

Substituting, we see

$$B_{k+1} = 4a - 2(4b)$$

$$B_{k+1} = 4(a - 2b)$$

Thus $4 | B_{k+1}$ by defn of divides.

Thus by induction

$$\forall n \in \mathbb{N}, 4 | B_n \quad \square$$

Why is it ok to assume strong inductive hypothesis?

Prove Prof. Reitz loves all natural numbers with these facts:

Fact 1 Prof. Reitz loves the number 1

Fact 2 If Prof Reitz loves all natural numbers from 1 up to K, then he loves $K+1$

Does Prof. Reitz love 1? Yes, Fact 1.

Does " love 2? Yes, Fact 2, $K=1$.
(I love 1)

Does " love 3? Yes, Fact 2, $K=2$
(I love 1, I love 2)

Does " love 4? Yes, Fact 2, $K=3$

⋮

Does " $1 \mid n$? Yes; Fact 2, $k = n-1$

Theorem NT 5.1: Every natural number $n > 1$ is either prime or divisible by a prime.

Proof (strong induction)

Base case $n=2$, 2 is prime so the theorem holds for $n=2$.

Inductive case. Suppose $k \in \mathbb{N}$, $k \geq 2$.

Ind. Hyp. (Suppose the theorem is true for all natural numbers from 2 to k.

Consider $k+1$.

Case 1 $k+1$ is prime, thus the theorem holds for $k+1$ ☒

Case 2 $k+1$ is not prime.

Then $d \mid k+1$ for some $d \in \mathbb{N}$, $d \neq 1$ and $d \neq k+1$.

Since $d \neq k+1$, $d < k+1$

So $d \le k$

Thus by inductive hyp, d is either prime or divisible by a prime.

If d is prime, we are done (a prime divides $k+1$).

If d is divisible by a prime,

$p \mid d$ for some prime, so $d = pa$, $a \in \mathbb{Z}$

also $d \mid k+1$, so $k+1 = db$, $b \in \mathbb{Z}$

$k+1 = (pa)b = p(ab)$ so $p \mid k+1$

Thus $k+1$ is either prime or divisible by a prime. $\square$

Prime: a natural number n, $n \geq 2$, is prime if the only positive divisors are 1 and n.