Exam #3 review

$P(n)$

#14 $\forall n \in \mathbb{Z}$, where $n \geq 0$, we have $3 \mid n^3 + 5n + 6$

Proof (Induction)

Base case if $n = 0$, then $n^3 + 5n + 6 = 6$
and $3 \mid 6$
thus the proposition is true for $n = 0$ $\square$

Inductive case Suppose $k \in \mathbb{N}$, $k \geq 0$,
and suppose $3 \mid k^3 + 5k + 6$

So $k^3 + 5k + 6 = 3a$, $a \in \mathbb{Z}$, by defn of divides

$+3k^2 + 3k + 6$ $\quad$ $+3k^2 + 3k + 6$

$k^3 + 3k^2 + 8k + 12 = 3a + 3k^2 + 3k + 6$

$k^3 + 3k^2 + 8k + 12 = 3(a + k^2 + k + 2)$

let $b = a + k^2 + k + 2$, $b \in \mathbb{Z}$

$k^3 + 3k^2 + 8k + 12 = 3b$

$(k+1)^3 + 5(k+1) + 6 = 3b$, $b \in \mathbb{Z}$

Thus $3 \mid (k+1)^3 + 5(k+1) + 6$ by defn of divides.

$\square$

Thus by induction,
$\forall n \in \mathbb{Z}, n \geq 0$, $3 \mid n^3 + 5n + 6$ $\blacksquare$

Scratch:

$1 \cdot k^3 + 3k^2 + 3k + 1 + 5k + 5 + 6 = 3b$

$\binom{3}{3}$ $\quad \binom{3}{2}$ $\quad \binom{3}{1}$ $\quad \binom{3}{0}$

$\frac{3!}{2! \cdot 1!} = 3$

$\longrightarrow n^3+3n^2+8n+12=36$

$\sqrt{15}$ is irrational

Ex: Prop $\sqrt{13}$ is irrational.

Proof (contradiction). Suppose $\sqrt{13}$ is rational

So $\sqrt{13}=\frac{a}{b}$, $a,b\in\mathbb{Z}$, $b\neq 0$, and wLOG assume a and b have no common factors.

So $13=\frac{a^2}{b^2}$

$13b^2=a^2$, note $b^2\in\mathbb{Z}$

$15b^2=a^2$

$3\cdot 5 \quad 3\cdot 5b^2=a^2$

$3|a^2$

$3|a$ Euclid's lemma

want $13|a$

So $13|a^2$ by defn of divides

So $13|a$ by Euclid's Lemma.

WARNING: ONLY works for primes: "prime$|a^2$"

thus $a=13n$, $n\in\mathbb{Z}$

$13b^2=(13n)^2$

$\frac{13b^2}{13}=\frac{13^2\cdot n^2}{13}$

$b^2=13\cdot n^2$, $n^2\in\mathbb{Z}$

Thus $13|b^2$

so by Euclids Lemma,

$13|b$.

F..for

So a, b have a common factor,
CONTRADICTION $\square$

Induction, part 2 (sums).

how many odd numbers?

for $n \in \mathbb{N}$ "the sum of the first n odd natural numbers"

n	sum of first n odd natural number
1	1
2	$1+3=4$
3	$1+3+5=9$
4	$1+3+5+7 = 16$
5	$1+3+5+7+9 = 25$
6	$+ 11 = 36$
7	$= 49$
k	$= k^2$

Proposition. $\forall n \in \mathbb{N}$, the sum of the first n odd natural numbers is equal to n^2.

Σ

Prop $\forall n \in \mathbb{N}$, $1+3+5+7+9+\ldots+(2n-1)=n^2$ — $P(n)$

$\forall n \in \mathbb{N}\; P(n)$

FACT: all these represent the same thing mathematically

$1+3+5+\ldots+117$

$1+3+5+7+\ldots+117$

$1+3+5+7+\ldots+115+117$

$1+3+5+7+9+11+13+\ldots+111+113+115+117$

Proof (Induction)

Base case $n=1$, left hand side $=1$ $P(1)$

right hand side $=1^2=1$ $\square$

Inductive case. Suppose $k \in \mathbb{N}$ $P(k)$

and $1+3+5+\ldots+(2k-1)=k^2$

$1+3+5+7+\ldots+(2k-3)+(2k-1)=k^2$

$+2k+1 \qquad +2k+1$

$1+3+5+7+\ldots+(2k-1)+(2k+1)=k^2+2k+1$

$1+3+5+7+\ldots+(2k-1)+(2k+1)=k^2+2k+1$

Thus $1+3+5+\ldots+(2(k+1)-1)=(k+1)^2$ $\square$ $P(k+1)$

thus by induction,

$\forall n \in \mathbb{N}$ $\quad 1+3+5+\ldots+(2n-1)=n^2$

$\square$

$2n-1 = 10-1 = 9$

$n = 5$

$1+3+5+7+9 = 5^2$

$+11 \qquad +11$

$2k+1 = 10+1 = 11$

$1+3+5+7+9+11 = 5^2+11$

$k+1 \; \cancel{5} = 6$