Induction

Q: What kind of statements can be proved by induction?

A: Universal statements about the natural numbers.

$$\forall n \in \mathbb{N},\ P(n)$$

Ex: $\forall n \in \mathbb{N},\ (n+2)^2 = n^2 + 4n + 4$ ("Any Algebra")

Two big ideas:

1. How does induction work?
2. What are the details/steps?

Example: Question - which natural numbers does Prof. Reitz love?

Notation: $P(n)$: "Prof. Reitz loves the number n"

We will accept these facts for now without proof.

Two Facts to use, to answer the question.

F1. Prof. Reitz loves the number 1. $\boxed{P(1)}$

F2. If Prof. Reitz loves a natural number k then he also loves $k+1$. $\boxed{\forall k \in \mathbb{N},\ P(k) \to P(k+1)}$

$P(n+1)$: "Prof. Reitz loves $n+1$"

	ANSWER	REASON
Do I love 1?	Y	Fact 1
Do I love 2?	I love 1	Fact 1
	I love $1+1=2$	fact 2 applied to 1
Do I love 3?	I love 1	F1
	I love $1+1=2$	F2 (to 1)
	I love $2+1=3$	F2 (to 2)

.

⋮

Do I love 10? I love 1 F1

I love 1+1=2 F2 (to 1)

I love 2+1=3 F2 (to 2)

I love 3+1=4 F2 (to 3)

I love 4+1=5 F2 (to 4)

⋮

I love 8+1=9 F2 (to 8)

I love 9+1=$\boxed{10}$ F2 (to 9)

Do I love 127? Yes use F1, F2

once. a lot of times

In fact, I love <u>all</u> natural numbers. $\boxed{\forall n \in \mathbb{N}, P(n)}$

Observation: Facts 1 and 2 are sufficient to conclude I love <u>all</u> natural numbers.

$P(\mathbb{N})$

$P(1,2,3,\dots)$

$P(1) \wedge P(2) \wedge \dots$

$\forall n \in \mathbb{N}, P(n)$

Given Fact 1 and Fact 2 → Conclude "Prof. Reitz loves every natural #"

Given

$P(1)$

$\forall n \in \mathbb{N}, P(k) \to P(k+1)$

→ Conclude

$\forall n \in \mathbb{N}, P(n)$

Proof by induction

Prop. $\forall n \in \mathbb{N}, P(n)$

Proof (Induction)

two steps: step 1 - "base step / base case"

[Prove $P(1)$]

step 2 - inductive step

[Prove $\forall n \in \mathbb{N}, P(k) \to P(k+1)$]

Thus by induction $\forall n \in \mathbb{N}, P(n)$.

Example

Proposition. $\forall n \in \mathbb{N}, 4 \mid 5^n - 1$

Scratch: $n=1$? $4 \mid 5^1 - 1$

$4 \mid 4$ ✓

T/F

$4 \mid 0$ Yes

$n=2$ $4 \mid 5^2 - 1$

$4 \mid (25-1)$

$4 \mid 24$ ✓ $24 = 4 \cdot 6$

$n=3$ $4 \mid 5^3 - 1$

$\forall n \in \mathbb{N}, P(n)$

$\boxed{P(n): 4 \mid 5^n - 1}$

$4 \mid 124$ ✓

Proof (induction)

base step $n=1$; goal: $4 \mid 5^1 - 1$ — $P(1)$

$5^1 - 1 = 5 - 1 = 4$,

and $4 \mid 4$,

so $4 \mid 5^1 - 1$, thus $P(1)$ holds ✓ $\square$

Inductive step

$\forall k \in \mathbb{N},\ P(k) \rightarrow P(k+1)$

$\forall k \in \mathbb{N},\ (4 \mid 5^k - 1) \rightarrow (4 \mid 5^{k+1} - 1)$

Suppose $k \in \mathbb{N}$, and $4 \mid 5^k - 1$

$\vdots$

Thus $4 \mid 5^{k+1} - 1$

Suppose $k \in \mathbb{N}$ and $4 \mid 5^k - 1$

so $5^k - 1 = 4a$, $a \in \mathbb{Z}$ by defn of divides.

multiplying by 5 on both sides

$$5 \cdot (5^n - 1) = 5 \cdot 4a$$

$$5^{n+1} - 5 = 20a$$

$$+4 \qquad +4$$

$$5^{n+1} - 1 = 20a + 4 = 4(5a+1)$$

let $b = 5a+1$, $b \in \mathbb{Z}$ by closure.

$$5^{n+1} - 1 = 4 \cdot b, \text{ where } b \in \mathbb{Z}$$

Thus $4 \mid 5^{n+1} - 1$ by defn of divides.

$\square$ (inductive step)

Thus by induction,

$\forall n \in \mathbb{N}$, $4 \mid 5^n - 1$

$\square$

Example ① state what is $P(n)$

② what is $P(1)$

fill in blank:

3

<u>induction step</u> Suppose ______

Thus ______ 4

<u>Proposition</u> If n is any natural number, then $5 \mid (n^5 - n)$

$\forall n \in \mathbb{N}, \ 5 \mid (n^5 - n)$

$P(n): 5 \mid n^5 - n$

$P(1): 5 \mid 1^5 - 1$

<u>Inductive step</u>

Suppose $k \in \mathbb{N}$ and $5 \mid k^5 - k$

$k^5 - k = 5a \qquad a \in \mathbb{Z}$.

$$(k+1)^5-(k+1)=5b,\ b\in\mathbb{Z}$$

Thus $5 \mid (k+1)^5-(k+1)$

$$(k+1)^5-(k+1)=$$

$$(k+1)^5-k-1=$$

$$(k+1)(k+1)(k+1)(k+1)(k+1)\ \underline{-k-1}=$$

$$\boxed{1\cdot k^5+5k^4+10k^3+10k^2+5k+\cancel{1}-k-\cancel{1}}$$

$$\binom{5}{5}\quad \binom{5}{4}\quad \binom{5}{3}\quad \binom{5}{2}\quad \binom{5}{1}\quad \binom{5}{0}$$

$$\binom{5}{3}={}_5C_3=\frac{5!}{(5-3)!\cdot 3!}$$

$5!$ ...

$$= \frac{5!}{2!\,3!} = \frac{5\cdot 4\cdot \cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{2\cdot 1\,\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

$$= 10$$

$$\boxed{n^5 + 5n^4 + 10n^3 + 10n^2 + 4n}$$

$$+5n^4 + 10n^3 + 10n^2 + 4n$$

$$n^5 - n = 5a \quad \ast$$

$$+5n^4 + 10n^3 + 10n^2 + 5n \quad \Big\} +5n^4 + 10n^3 + 10n^2 + 5n$$

$$(n+1)^5 - (n+1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 4n = 5a + 5n^4 + 10n^3 + 10n^2 + 5n$$

$$(n+1)^5 - (n+1)$$

$5b$?

$$= 5\underbrace{(a + n^4 + 2n^3 + 2n^2 + n)}_{b \in \mathbb{Z} \text{ by closure}}$$

$$(n+1)^5 - (n+1) = 5b, \quad b \in \mathbb{Z}$$

thus $\quad 5 \mid (n+1)^5 - (n+1)$