# Day 21

Chapter 9: Disproof

| | |
|---|---|
| - conjecture | - disproof<br>- counterexample |

**Example.** Is each proposition true or false?

1. Every even natural number is the sum of two odd natural numbers.
2. Every even natural number is the sum of two perfect squares.
3. Every even natural number is the sum of two primes.

THREE TYPES OF STATEMENTS:

| **Known to be true** (Theorems & propositions) | **Truth unknown** (Conjectures) | **Known to be false** |
|---|---|---|
| Examples:<br>• Pythagorean theorem<br>• Fermat's last theorem (Section 2.1)<br>• The square of an odd number is odd.<br>• The series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. | Examples:<br>• All perfect numbers are even.<br>• Any even number greater than 2 is the sum of two primes. (Goldbach's conjecture, Section 2.1)<br>• There are infinitely many prime numbers of form $2^n - 1$, with $n \in \mathbb{N}$. | Examples:<br>• All prime numbers are odd.<br>• Some quadratic equations have three solutions.<br>• $0 = 1$<br>• There exist natural numbers $a, b$ and $c$ for which $a^3 + b^3 = c^3$. |

**Definitions & Theorems**

- Definition. A statement whose truth is unknown is called a **conjecture.**
- Definition. When we prove a statement P is false, we call this a **disproof** of P.

**How to Disprove P:** Prove $\sim P$ .

**CHEAT SHEET:** How to disprove a statement of the form...

- To disprove $\forall x\, P(x)$, give an example of an $x$ that makes $P(x)$ false (such an $x$ is called a **counterexample**).
- To disprove $P(x) \Longrightarrow Q(x)$, give an example of an $x$ that makes $P(x)$ true but $Q(x)$ false.
- To disprove $\exists x\, P(x)$, prove the statement $\forall x,\ \sim P(x)$.
- To disprove P by contradiction, assume P is true and deduce a contradiction $C \wedge \sim C$ .

**Prove or disprove each conjecture.**

Conjecture. For every $n \in \mathbb{Z}$, the integer $f(n) = n^2 - n + 11$ is prime.

Conjecture. If A, B and C are sets, then $A - (B \cap C) = (A - B) \cap (A - C)$ .

Conjecture. If A and B are sets, then $P(A) - P(B) \subseteq P(A - B)$ .

Conjecture. For every $n \in \mathbb{Z}$, the integer $f(n) = n^2 - n + 11$ is prime.

Is it true or false?

$n = 3 \quad f(n) = 3^2 - 3 + 11 = 9 - 3 + 11$

$= 6 + 11 = 17$

$n=11$ $\quad f(11) = 11^2 - 11 + 11 = 11^2 = 121$
$= 11 \cdot 11$

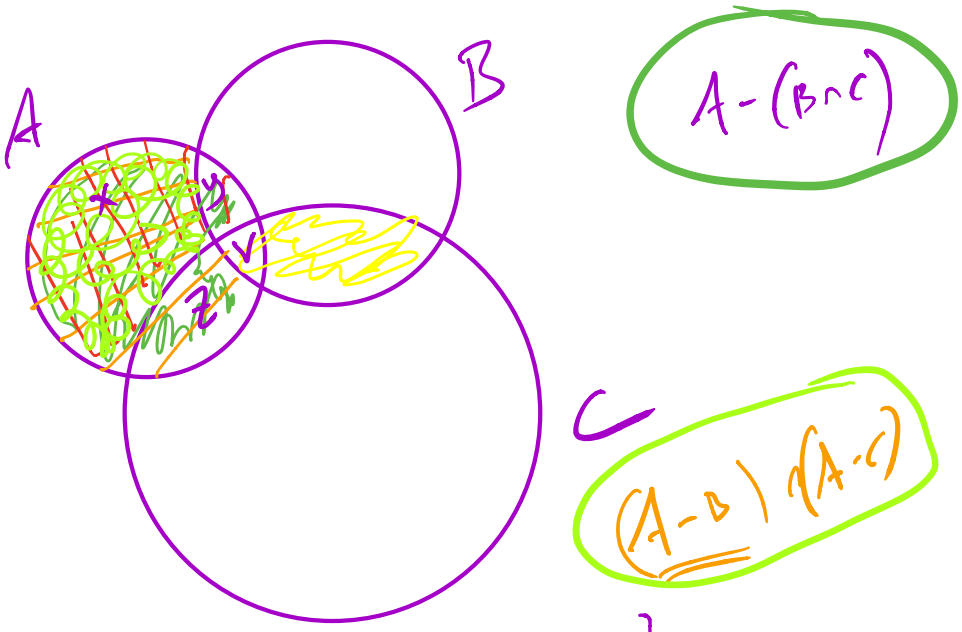
NOT PRIME.

<u>Disproof of Conjecture</u>: if $n=11$, then $f(n)=121$ which is not prime. Thus the conjecture is false $\square$

$n=4$ $\quad f(4) = 4^2 - 4 + 11$
$= 16 - 4 + 11 = 23$

$2^{(2^n)} - 1$

Conjecture. If A, B and C are sets, then $A - (B \cap C) = (A - B) \cap (A - C)$.




$A - (B \cap C)$

$(A - B) \cap (A - C)$

<u>False</u>

$A = \{x, y, z, v\}$
$B = \{y, v\}$
$C = \{v, z\}$

<u>Disproof</u> Let $A = \{x, y, z, v\}$
$B = \{y, v\}$
$C = \{v, z\}$

Then $A - (B \cap C) = \{x, y, z\}$

and $(A-B)\cap(A-C) = \{x,z\} \cap \{x,y\}$

Thus $A-(B\cap C) \neq (A-B)\cap(A-C)$ $\square$