Day 20

Chapter 8: Proofs involving sets

Vocabulary

- Proof that $x \in A$	- Proof that $A \subseteq B$ - Proof that $A = B$

Review of Sets: Main definitions from Chapter 1

- Defn. $A \times B = \{ (x, y) : x \in A, y \in B\}$
- Defn. $A \cup B = \{ x : (x \in A) \vee (x \in B)\}$
- Defn. $A \cap B = \{ x : (x \in A) \wedge (x \in B)\}$
- Defn. $A - B = \{ x : (x \in A) \wedge (x \notin B)\}$
- Defn. $\overline{A} = U - A$
- Defn. $P(A) = \{X : X \subseteq A\}$

How to prove $a \in A$.

Show that the given object a satisfies the definition of the set A.

Example. a. If $a = \{2, 12, 17\}$ and $A = \{X \in P(\mathbb{N}) : |X| = 3\}$, then prove $a \in A$.

b. If $C = \{3x^3 + 2 : x \in \mathbb{Z}\}$, then prove $-22 \in C$.

How to prove $A \subseteq B$

To prove $A \subseteq B$, *prove the statement: if* $a \in A$, *then* $a \in B$.

Direct Proof	**Contrapositive Proof**
Proof. Suppose $a \in A$. … Therefore $a \in B$. Thus $a \in A$ implies $a \in B$, and so $A \subseteq B$.	*Proof.* Suppose $a \notin B$. … Therefore $a \notin A$. Thus $a \notin B$ implies $a \notin A$, and so $A \subseteq B$.

Example 8.5. Prove that $\{x \in \mathbb{Z} : 18|x\} \subseteq \{x \in \mathbb{Z} : 6|x\}$

Example 8.8. Prove that if A and B are sets, then $P(A) \cup P(B) \subseteq P(A \cup B)$.

Example 8.9. Suppose that A and B are sets. If $P(A) \subseteq P(B)$ then $A \subseteq B$.

How to prove A=B
Proof.
[Prove $A \subseteq B$.]
[Prove $B \subseteq A$.]

Therefore, since $A \subseteq B$ and $B \subseteq A$, it follows that $A = B$.

Example 8.12. Given sets A, B and C, prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proofs About Sets

three types:

1. Prove $a \in B$.
2. $A \subseteq B$
3. $A = B$

Membership Proofs

Prop. $a \in B$.

Proof: [show that a satisfies the definition of the set B] $\square$

Example

Prop. If $a = \{2, 27, 312\}$, then
$$a \in \{X \in \mathcal{P}(\mathbb{N}) : |X| = 3\}.$$

Proof Note: $a \subseteq \mathbb{N}$ (since every member of a is a natural number).

so $a \in \mathcal{P}(\mathbb{N})$.

also $|a| = 3$.

Thus $a \in \{X \in \mathcal{P}(\mathbb{N}) : |X| = 3\}$.

$\square$

Prop. let $C = \{3x^2 + 2 : x \in \mathbb{Z}\}$.
then $-22 \in C$.

$x = 1$
$3 \cdot 1^2 + 2 = 5$
$5 \in C$

Proof. The statement is false, all elements of C are positive. $\square$

Prop. let $C = \{3x^3 + 2 : x \in \mathbb{Z}\}$.
then $-22 \in C$.

Proof. Let $x = -2$.

then $3(-2)^3+2 = 3(-8)+2 = -24+2$
$= -22.$

thus $-22 \in C.$ ☑

Digression - set operations (recall from Chapter 1)

A, B sets

$A \times B = \{(a,b) : a \in A, \text{ and } b \in B\}$

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

$A - B = \{x : x \in A \text{ and } x \notin B\}$

$\overline{A} = U - A = \{x \in U : x \notin A\}$

$P(A) = \{x : x \subseteq A\}$

UV

Subset Proofs

Prop. $A \subseteq B$

what we want to prove is the statement: if $a \in A$ then $a \in B$

Direct proof	Contrapositive Proof
Proof. Suppose $a \in A$	Proof. Suppose $a \notin B$

Proof Suppose $a \in A$	**Proof** ...
$\vdots$	$\vdots$
Thus $a \in B$.	Thus $a \notin A$
Therefore $A \subseteq B$ $\square$	Therefore $A \subseteq B$ $\square$

Example

Prop. $\{x \in \mathbb{Z} : 18 \mid x\} \subseteq \{x \in \mathbb{Z} : 6 \mid x\}$

Proof (Direct). Suppose $a \in \{x \in \mathbb{Z} : 18 \mid x\}$.

Thus $a \in \mathbb{Z}$ and $18 \mid a$.

So $a = 18b$ for some $b \in \mathbb{Z}$, by defn. of divides.

$a = 6(3b)$, note $3b \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under $\cdot$.

Thus $6 \mid a$ by defn of divides.

Thus $a \in \{x \in \mathbb{Z} : 6 \mid x\}$ $\square$

Prop If A, B are sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

Proof Suppose A, B are sets

Suppose $c \in \mathcal{P}(A) \cup \mathcal{P}(B)$

Thus $c \in \mathcal{P}(A)$ or $c \in \mathcal{P}(B)$ by defn of $\cup$.

WLOG suppose $c \in \mathcal{P}(A)$.

thus $c \subseteq A$ by defn of $\mathcal{P}(A)$.

$A \subseteq A \cup B$ by defn of $\cup$

therefore $c \subseteq A \cup B$

$C \subseteq A \cup B$

Thus $C \in \mathcal{P}(A \cup B)$ by defn of $\mathcal{P}(A \cup B)$. $\square$

~~Case 2: $C \in \mathcal{P}(B)$~~
~~thus $C \subseteq B$~~
~~$B \subseteq A \cup B$~~
~~therefore $C \subseteq A \cup B$~~
~~$C \in \mathcal{P}(A \cup B)$~~

Set Equality

Prop $A = B$

Proof [prove $A \subseteq B$]
[prove $B \subseteq A$]
Thus $A = B$ $\square$

Proposition Suppose A, B and C are sets, then $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof ($\subseteq$) suppose $\cancel{p}$ $(q, r) \in A \times (B \cap C)$

then $p = (q, r)$, $q \in A$, $r \in B \cap C$
by defn of ordered pair.

since $r \in B \cap C$, in particular $r \in B$ and $r \in C$
by defn of $\cap$.

since $q \in A$ and $r \in B$, then $(q,r) \in A \times B$
by defn of $\times$.

since $r \in C$, then $(q,r) \in A \times C$

and

Thus $(q,r) \in (A \times B) \cap (A \times C)$ by defn of $\cap$. $\square$

$(\supseteq)$ Suppose $p \in (A \times B) \cap (A \times C)$

then $p \in A \times B$ and $p \in A \times C$ by defn $\cap$.

since $p \in A \times B$, $p = (q, r)$ $q \in A$ and $r \in B$
by defn $\times$

$(q,r) \in A \times C$ so $q \in A$ and $r \in C$

Thus $r \in B \cap C$ by defn $\cap$.

Since $p \in A$ and $r \in B \cap C$ by defn of $\times$

then $(p, r) \in A \times (B \cap C)$

$\square$