Prop Every even natural number can be written as the sum of two odd numbers

$n = 10$. $3+7$, $1+9$, $5+5$.

$n = 12$ $1+11$, $5+7$, $9+3$, etc...

$n = 106{,}378$ $106377 + 1$

$106375 + 3$

etc...

TRUE.

Proof (Direct) Suppose n is an even natural number.

$n-1$ is odd (by the division algorithm).

1 is odd

and $(n-1)+1 = n$ $\square$

Prop. Every even natural number can be written as the sum of two perfect squares.

$n = 14$

Perfect squares $\{0, 1, 4, 9, 16, 25, \ldots\}$

a counter example.

FALSE

a conjecture

"Goldbach Conjecture"

Prop. Every even natural number (except 2) can be written as the sum of two primes.

Primes: $\{2, 3, 5, 7, \cancel{9}, 11, 13, 17, 19, 23, \ldots\}$

$n = 18$ $\quad 7 + 11$

$n = 26$ $\quad 7 + 19, \; 13 + 13$

$n = 28$ $\quad 23 + 5$

$n = 36$ $\quad 29 + 7$

Proof (direct) Suppose $n \in \mathbb{N}$, n is even, $n \neq 2$.

UNKNOWN whether true or false.

$\mathbb{N} = \text{Primes} \cup \text{Composite} \cup \{1\}$

two new types of proof:

- If-and-only-if (biconditional) statements

- Existential statements

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conditional $P \to Q$

Methods of proof
- Direct
- contrapositive
- contradiction

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biconditional $P \leftrightarrow Q$

Recall $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$

To prove $P \leftrightarrow Q$

Prop. $P \leftrightarrow Q$

Proof. [step 1: "in the forward direction..." $P \to Q$, using direct, contrapositive, or contradiction]

"conversely," [step 2: $Q \to P$, using direct, contrapositive, contradiction] the converse.

$\square$

$a \in \mathbb{N}$, if $6 | a$ then $2 | a$

$P \to Q$

$Q \to P$

$P \leftrightarrow Q$

Prop Suppose $x \in \mathbb{Z}$. x is odd P if and only if x^2 is odd. Q

Proof. In the forward direction, suppose $x \in \mathbb{Z}$ and x is odd.

$P \to Q$

Suppose P

So $x = 2n+1$, $n \in \mathbb{Z}$ by defn of odd.

$$x^2 = (2n+1)^2$$
$$x^2 = 4n^2 + 4n + 1$$
$$x^2 = 2(2n^2+2n) + 1, \quad \text{(algebra)}$$

note $2n^2+2n \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under $+, \cdot$.

Thus Q

Thus x^2 is odd, by defn of odd.

optional [Therefore if x is odd then x^2 is odd].

if x^2 is odd then x is odd

Conversely, suppose x is an even integer.

$Q \to P$

suppose $\sim P$

thus $x = 2m$, $m \in \mathbb{Z}$ by defn even

so $x^2 = (2m)^2$

$$x^2 = 4m^2 = 2(2m^2)$$

Thus $\sim Q$

$2m^2 \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under $\cdot$.

Thus x^2 is even, by defn of

even.

Therefore if x^2 is odd then x is odd.

Thus x is odd if and only if x^2 is odd $\square$

Existence proofs

"give an example"

Prop $\exists x\, P(x)$

Proof. [give an example of some $x = a$, show $P(a)$]

Prop. There exists an even prime number.

Proof. Consider the number 2.

2 is prime (it has divisors 1 and 2)

2 is even, since $2 = 2 \cdot 1$, $1 \in \mathbb{Z}$, by defn of even.

Thus 2 is an even prime number.