Day 17

Chapter 6

Vocabulary

- contradiction	- proof by contradiction

Definitions & Theorems

- Definition. A statement which cannot be true (all rows in the truth table are False) is called a **contradiction**.
- Definition. A real number x is **rational** if $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$.
  A real number x is **irrational** if it is not rational, that is if $x \neq \frac{a}{b}$ for every $a, b \in \mathbb{Z}$.
- Theorem. Every natural number greater than 1 has a unique factorization into primes.
- Theorem. **(Euclid's Lemma)**. Suppose p is a prime and $a, b \in \mathbb{Z}$. If $p|ab$, then $p|a$ or $p|b$.

Proposition. If $a, b \in \mathbb{Z}$ then $a^2 - 4b \neq 2$.

Outline for Proof by Contradiction

Proposition. P.

Proof. Suppose $\sim P$.

…

Therefore $C \wedge \sim C$.

Theorem. The number $\sqrt{2}$ is irrational.

Theorem. The number $\sqrt{7}$ is irrational.

Theorem. The following numbers are irrational:

a) $\sqrt{15}$ b) $\sqrt{21}$ c) $\sqrt{12}$ d) $\sqrt{18}$

Notes on Using Proof by Contradiction

- To prove a statement of the form $P \Rightarrow Q$ using contradiction:
  Start by assuming $P \wedge \sim Q$ (and then prove $C \wedge \sim C$).
- To prove a statement of the form $\forall x P(x)$ using contradiction:
  Start by assuming $\exists x \sim P(x)$ (and then prove $C \wedge \sim C$).

$\boxed{P: \text{if } a,b \in \mathbb{Z} \text{ then } a^2 - 4b \neq 2}$

Proposition. If $a, b \in \mathbb{Z}$ then $a^2 - 4b \neq 2$.

$a = 4$ $16 - 4b =$ 0 or 4 but not 2.

$a = 7$ $49 - 4b \neq 2$

what if the proposition is false? what goes wrong?

negation of our proposition.

Proof Suppose $a, b \in \mathbb{Z}$ and $a^2 - 4b = 2$

$+4b$ $+4b$

$a^2 = 2 + 4b$

$a^2 = 2(1+2b)$

NOTE $1+2b \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under $+, \cdot$

Thus a^2 is even.

So a is even (stated in class)

Theorem: if $x \in \mathbb{Z}$ and x^2 is even, then x is even.

$a = 2n$, $n \in \mathbb{Z}$ by defn of even.

Substituting, we have

$(2n)^2 = 2(1+2b)$

$\frac{4n^2}{2} = \frac{2(1+2b)}{2}$

$2n^2 = 1+2b$

$-2b \qquad -2b$

$2n^2 - 2b = 1$

$2(n^2 - b) = 1$

Note $n^2 - b \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under $-, \cdot$

so 1 is even, by defn of even.

BUT we know 1 is not even.

CONTRADICTION $\square$

C: 1 is even

$\sim C$: 1 is not even

~~Or we say~~

~~C: $a^2 - 4b = 2$~~

~~$\sim C$: $a^2 - 4b \neq 2$~~

Proof By Contradiction

Proposition. P

Proof (contradiction) Suppose $\sim P$

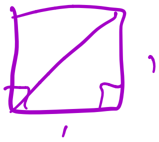
$\vdots$

$C \wedge \sim C$

Thus, by contradiction, P $\square$

Recall x is rational if $x = \frac{a}{b}$, $a, b \in \mathbb{Z}$, and $b \neq 0$.

Defn: $x \in \mathbb{R}$ is irrational if x is not rational.



Prop. $\sqrt{2}$ is irrational.

Proof (contradiction) Suppose $\sqrt{2}$ is rational.

Then $\sqrt{2} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$, by defn of rational.

Without loss of generality, a, b have no common factors, other than 1

(OR $\frac{a}{b}$ is in lowest terms).

$(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$ (algebra)

$b^2 \cdot 2 = \frac{a^2}{b^2} \cdot b^2$

$2b^2 = a^2$

Note $b^2 \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under $\cdot$.

thus a^2 is even, by definition of even.

thus a is even (stated in class)

So $a = 2n$, $n \in \mathbb{Z}$ defn of even

substituting, we have

Thm P

Proof. Suppose $\sim P$

⋮

P

$\frac{3}{2} = \sqrt{2}$?

test $\left(\frac{3}{2}\right)^2 = \frac{9}{4} \neq 2$

$\frac{1414}{1000} = \sqrt{2}$

$\left(\frac{1414}{1000}\right)^2$

$= \frac{1414^2}{1000^2}$

$\frac{4}{6} = \frac{10}{15}$?

$= \boxed{\frac{2}{3}} = \frac{8}{12} = \frac{100}{150}$

"in lowest terms" means

Defn a rational number $\frac{a}{b}$ is in

$$2b^2 = (2n)^2$$

$$\frac{2b^2}{2} = \frac{4n^2}{2}$$

lowest terms if a,b do not have any common divisors except $1,-1$.

$$b^2 = 2n^2$$

Since $n^2 \in \mathbb{Z}$ by closure, b^2 is even.

Thus b is even (stated in class)

So a and b are even

So 2 is a common factor of a,b / so $\frac{a}{b}$ is not in lowest terms

Contradiction.

Thus $\sqrt{2}$ is irrational. $\square$

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try similar propositions:

$\sqrt{7}$ is irrational

$\sqrt{15}$ is irrational

$\sqrt{21}$ is irrational

$\vdots$ etc

$17 + \sqrt{2} - \sqrt{2}$

17

<u>Prop</u> $\sqrt{7}$ is irrational.

<u>Proof (contradiction)</u> Suppose $\sqrt{7}$ is rational.

Then $\sqrt{7} = \frac{a}{b}$ for some $a,b \in \mathbb{Z}$, $b \neq 0$

WLOG, $\frac{a}{b}$ is in lowest terms

$$(\sqrt{7})^2 = \left(\frac{a}{b}\right)^2$$

$$b^2 \cdot 7 = \frac{a^2}{b^2} \cdot b^2$$

$$7b^2 = a^2$$

$b^2 \in \mathbb{Z}$ by closure of $\mathbb{Z}$...

So $7 \mid a^2$ $\quad 7 \mid a \cdot a$

thus $7 \mid a$ or $7 \mid a$ by Euclids lemma.

So $7 \mid a$

$\boxed{\text{want: } 7 \mid a}$

Next steps:

substitute $a = 7n$

show $7 \mid b^2$

thus $7 \mid b$

contradiction. $\frac{a}{b}$ is not in lowest terms since 7 is common divisor $\square$

Recall: Euclids Lemma.

if p prime, $a, b \in \mathbb{Z}$, and $p \mid ab$ then $p \mid a$ or $p \mid b$