Exam 4 Review

The exam will cover sections 11.1 -- 11.7 (NOTE: 11.7 is simply an "overview section", which summarizes the results of 11.1--11.6). The answer key follows the problems. If you find any errors or have any questions, please post them on the OpenLab or send me an email: jreitz@citytech.cuny.edu

1. What is the difference between a sequence and a series?
2. For each statement below, decide whether it is true or false and *explain your answer*.
   a. If a sequence $\{a_n\}$ converges, then the series Σa_n also converges.
   b. If a series Σa_n converges, then the sequence $\{a_n\}$ converges to zero.
   c. If a sequence $\{a_n\}$ converges to zero, then the series Σa_n converges.
3. Find a formula for a_n in each sequence (assume that the first term has $n=1$).
   a. $6, -4, \frac{8}{3}, -\frac{16}{9}, \frac{32}{27}, \ldots$
   b. $\frac{2}{5}, \frac{3}{25}, \frac{4}{125}, \frac{5}{625}, \ldots$
   c. $22, 16, 22, 16, 22, 16, \ldots$
4. For each sequence, determine whether it converges. If it converges, find the limit.
   a. $\left\{\frac{1}{n^3}\right\}$ b. $a_n = \frac{n}{n+1}$ c. $\frac{2n^2+3}{5n^2+3n+2}$
   d. $\left\{\frac{n+\ln n}{n}\right\}$ e. $\frac{(-1)^n}{n^2}$ f. $a_n = \frac{5n!}{7n^n}$
5. Find the sum:
   a. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$
   b. $5 - \frac{10}{7} + \frac{20}{49} - \frac{40}{343} + \ldots$
   c. $\left(\frac{5}{8} - \frac{5}{9}\right) + \left(\frac{5}{9} - \frac{5}{10}\right) + \left(\frac{5}{10} - \frac{5}{11}\right) + \left(\frac{5}{11} - \frac{5}{12}\right) + \ldots$
6. Find the sum:
   a. $\sum_{n=1}^{\infty} \frac{1}{11^n}$ b. $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^{n-1}}{3^{n+2}}$ c. $\sum_{n=1}^{\infty} \frac{8}{n(n+2)}$ d. $\sum_{n=1}^{\infty} \frac{3^n + 14^n}{17^n}$
7. Given the repeating decimal $x = 71.7435435435435\ldots$,
   a. Express x as the sum of a finite decimal and a geometric series.

b. Express x as a ratio of integers.

8. Determine whether each series converges (show your work, and indicate any tests you used).

a. $\sum_{n=1}^{\infty} \frac{1}{(2n+3)^4}$ b. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

9. Determine whether each series converges (show your work, and indicate any tests you used).

a. $\sum_{n=1}^{\infty} \frac{2n^2+7}{3n^4+2n^2-5}$ b. $\sum_{n=1}^{\infty} \frac{1}{3^n-1}$ c. $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$

d. $\sum_{n=1}^{\infty} \frac{5n^5+3n}{\sqrt{n^{11}+n^6+2}}$ e. $\sum_{n=1}^{\infty} \frac{\sin^2 n+1}{2^n}$

10. Determine whether each series converges (show your work, and indicate any tests you used).

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$ b. $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2n-7}{5n+1}$ c. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

11. Determine whether each series is absolutely convergent, conditionally convergent, or divergent (show your work, and indicate any tests you used).

a. $\sum_{n=1}^{\infty} \frac{(-6)^n}{n!}$ b. $\sum_{n=1}^{\infty} \frac{(-n)^5}{4^n}$ c. $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2 \cdot 2^n}{n!}$

d. $\sum_{n=1}^{\infty} \left(\frac{n^3+7x}{5n^3+3x^2-7x+1}\right)^n$

Answer Key

1. *Hint: I do not need a lot of technical language or long explanations -- the difference is straightforward and shouldn't take too much time to state. If you're having trouble with this question, try to answer these: What exactly is a sequence (in your own words)? What is a series?*
2. *Hints:*
   a. Consider a sequence that converges to some nonzero number, like the sequence 2, 2, 2, 2,.... Does the corresponding series 2+2+2+2... also converge?
   b. The Test for Divergence contains the answer. If we are adding up numbers, and the answer is approaching a limit (instead of growing towards infinity), what can we say about the numbers we are adding up?
   c. This is tempting, but false. There is a famous counterexample that we have discussed a number of times, in which the terms of the sequence go to zero, but the series diverges. Which series is it? (Hint: it's related to music)
3. a. $a_n = 6 \cdot \left(\frac{-2}{3}\right)^{n-1}$ b. $a_n = \left(\frac{n+1}{5^n}\right)$ c. $a_n = 19 + 3 \cdot (-1)^{n-1}$
4. a. Converges, $\lim_{n \to \infty} \frac{1}{n^3} = 0$
   b. Converges, $\lim_{n \to \infty} \frac{n}{n+1} = 1$ (multiply top & bottom by $\frac{1}{n}$)
   c. Converges, $\lim_{n \to \infty} \frac{2n^2+3}{5n^2+3n+2} = \frac{2}{5}$ (multiply top/bottom by $\frac{1}{n^2}$)
   d. Converges, $\lim_{n \to \infty} \frac{n + \ln n}{n} = 1$ (Use L'Hospital's Rule)
   e. $\lim_{n \to \infty} \frac{(-1)^n}{n^2} = 0$ (if the absolute value has limit = 0, then so does the original sequence)
   f. $\lim_{n \to \infty} \frac{5n!}{7n^n} = 0$ (use the squeeze theorem, with $0 \leq \frac{5n!}{7n^n} \leq \frac{5}{7n}$)
5. a. The sum is $\frac{1}{3}$. (geometric series with $a = \frac{1}{4}, r = \frac{1}{4}$)
   b. The sum is $\frac{35}{9}$. (geometric series with $a = 5, r = \frac{-2}{7}$)
   c. The sum is $\frac{5}{8}$. (telescoping series)

6.
a. $\frac{1}{10}$ (geometric series, $a=\frac{1}{11}, r=\frac{1}{11}$)

b. $-\frac{1}{45}$ (geometric series, $a=-\frac{1}{27}, r=\frac{-2}{3}$).

c. 6 (Use partial fractions to rewrite the series as $\sum_{n=1}^{\infty}\frac{8}{n(n+2)}=\sum_{n=1}^{\infty}\frac{4}{n}-\frac{4}{n+2}$. Write out the first few terms and notice that everything is cancelled except the terms 4 and 2, which give 6).

d. $\frac{205}{42}$ (split into two series and evaluate each separately - they are both geometric series)

7.
a. $x=71.7+\sum_{n=1}^{\infty}\frac{435}{10000}\cdot(\frac{1}{10^3})^{n-1}$

b. $\frac{119453}{1665}$

8.
a. Converges by the Integral Test (Hint: use substitution with $u=2x+3$, don't forget to convert bounds of integration to *u*)

b. Converges by the Integral Test (Hint: use substitution with $u=\frac{1}{n}$, don't forget to convert bound of integration to *u*)

9. *NOTE: your answer (converges/diverges) should be the same, but the particular method and series you use to establish it might differ.*

a. Converges, by the limit comparison test with $b_n=\frac{2n^2}{3n^4}=\frac{2}{3n^2}$ (which converges since it is a p-series with p>1)

b. Converges, by the limit comparison test with $b_n=\frac{1}{3^n}$ (which converges since it is a geometric series $r=\frac{1}{3}<1$)

c. Diverges, by the comparison test with the smaller series $b_n=\frac{1}{\sqrt{n}}$ (which diverges since it is a p-series with p<1)

d. Diverges, by the limit comparison test with $b_n=\frac{5n^5}{\sqrt{n^{11}}}=\frac{5}{n^{1/2}}$ (which diverges since it is a p-series with p<1)

e. Converges, by the comparison test with the larger series $b_n=\frac{2}{2^n}$ (which converges since it is a geometric series with $r=\frac{1}{2}<1$)

10.
a. Converges by the alternating series test, since $\lim_{n\to\infty}\frac{1}{2n+1}=0$.

b. Diverges by the Test for Divergence, since $\lim_{n\to\infty} \frac{2n-7}{5n+1} = \frac{2}{5} \neq 0$.

c. Converges by the alternating series test, since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

11. a. Absolutely convergent, by the ratio test.
    b. Absolutely convergent, by the ratio test.
    c. Absolutely convergent, by the ratio test.
    d. Absolutely convergent, by the root test.