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Developing "Real World" Classroom Examples and Case Studies

Introduction

We have developed a set of classroom examples and exercises to be introduced in Mat1175 (or/and Mat1275) as Math applications to solve "Real World" problems related with the fall of objects (parachutist, skydiver, raindrop, etc.) in presence of drag.

The application of drag formulae using Math techniques (learned in class) is not a simple substitution of numbers in formulae and Math manipulation to find the solution of the problems, but it is an active learning process on the use the Math models to solve and give answer to real-life problems with practical interest.

The applications are related and all together form a module that will be delivered in one session.

We create as well pre & post surveys with questions that will assess the students' reaction to the application module both in the affective and content/skills areas.

Note: The material (which is prepared to be presented in class by the instructor) is associated with notes and comments in order to help students better understand the physical phenomenon and the mathematical model.

1. Newton's Formula on the Resistance of Air

The magnitude R of the resistance of air (drag) that a falling object/body (parachutist, skydiver, etc.) encounters in air is given by Newton's formula:

$$R = \frac{c}{2} d A v^2 \qquad (1.1)$$

where:

- d is the density of air that depends on the altitude h of the object over the sea level. Near the ground the air density is $d = 1.205 kg/m^3$
- v is the speed of the falling body.
- A is the largest effective cross-sectional area of the object that is perpendicular to the falling direction.
- c is the drag coefficient of the body (called also the coefficient of air resistance).

The drag force R (resistance of air) acts on the object in the opposite direction of motion.

The drag coefficient c for relatively small speeds (till around $v = 250 m/s$) is considered to be a constant and can be determined experimentally. It is an aerodynamic characteristic of the falling body.

In the attached Appendix A, there are given the values of the coefficient of resistance for some regular objects.

The drag coefficient for the parachutist that falls with a deployed hemispherical parachute is $c = 1.33$ (see table in appendix A).

The drag coefficient of a parachutist in motion with closed parachute is $c = 0.8$, while the average cross-section area is $A = 0.5m^2$.

The Ground (Terminal) Speed of Falling Objects

(Parachutist, Skydiver, Raindrop, Hailstone, etc.)

A parachutist of mass m jumps from a hovering helicopter. Immediately after the parachutist leaves the helicopter, the speed of the parachutist "v" increases and, as the formula (1.1) shows, the drag force R increases.

At a certain moment the drag force R becomes equal to the force of gravity exerted on the parachutist, i.e. equal to the weight of the parachutist $P = mg$.

From that instant, the resultant force and the acceleration become and remain zero. As result the parachutist falls on the ground with a constant speed that can be determined by the equation:

$$R = P . \tag{1.2}$$

The equation (1.2) can be used to find the speed of the parachutist (See example 1.2).

$$v = \sqrt{\frac{2mg}{cdA}} . \tag{1.3}$$

Example 1.1 Resistance of Air (Optional)

A parachutist is launched from a hovering helicopter in absence of wind with a closed parachute.

a. Find the resistance of air R that is exerted on the parachutist/skydiver falling with closed parachute when the altitude of the parachutist is $h = 500m$, and his speed is $v = 50m/s$. The cross-section area of the parachutist body is $A = 0.5m^2$ while the coefficient of resistance is $c = 0.8$.

b. Find the resistance of air that is exerted on the parachutist/skydiver falling with fully deployed parachute considering that the diameter of a hemisphere parachute is $D = 8m$ and that the coefficient of resistance of a circular parachute is $c = 1.33$.
The speed of the parachutist is $v = 10m/s$. The altitude of the parachutist is $h = 500m$.

Note that at the altitude 500 meters over the sea level, the density of air is $d = 1.16kg/m^3$.

Solution

a. Substituting in (1.1) we find

$$R = \frac{c}{2} dAv^2 = \frac{(0.8)}{2}(1.16)\cdot(0.5)\cdot(50)^2 = 580N .$$

b. The cross-section area is equal to the circular area of the parachute with diameter $D = 8m$, i.e.

$$A = \pi \cdot D^2 / 4 = 3.14 \cdot 8^2 / 4 = 50.27 m^2 .$$

The resistance of air is

$$R = \frac{c}{2} dAv^2 = \frac{(1.33)}{2}(1.16)\cdot(50.27)\cdot(10)^2 = 3877.83N .$$

Example 1.2 Ground Speed of the Parachutist

A parachutist of mass $m = 100kg$ is launched from a hovering helicopter with a deployed parachute of diameter $D = 8m$. During the fall, on the parachutist there is exerted the force of gravity,

$$P = mg , \tag{1.4}$$

and the resistance of air,

$$R = \frac{c}{2} dAv^2 , \tag{1.5}$$

in opposite direction of the gravity force.

(a) Find the formula for the speed v of the parachutist during the fall
(b) Find the ground speed of the parachutist if:

- The density of air on the ground is $d = 1.205 kg / m^3$
- The cross-section of the parachute is $A = \pi \cdot D^2 / 4 = \pi(8)^2 / 4 = 50.27 m^2$.
- The coefficient of resistance of a hemisphere parachute is $c = 1.33$.
- Acceleration of the gravity force is $g = 9.81 m / s^2$.

Solution

(a) Substituting the quantities given by (1.4) and (1.5) in formula (1.2) we have:

$$\frac{c}{2} dAv^2 = mg \; . \tag{1.6}$$

Solving (1.6) for v^2 we obtain:

$$v^2 = \frac{2mg}{cdA} \tag{1.7}$$

Now, solving the second degree equation (1.7) for the speed v, we obtain the formula for the speed of the parachutist:

$$v = \sqrt{\frac{2mg}{cdA}} \tag{1.8}$$

Note that the only parameter that changes with altitude is the density of air d.

(b) Substituting in (1.8) the respective values given above we find that the ground speed of the parachutist is

$$v = \sqrt{\frac{2mg}{cdA}} = \sqrt{\frac{2(100)\cdot(9.81)}{(1.33)\cdot(1.205)\cdot(50.27)}} = 4.93 m/s \; . \tag{1.9}$$

Note: Formula (1.8) can be employed to estimate the ground speed of any object falling with relatively not great speeds.

Exercise 1.3 Parachute Accidents

(a) Find the ground speed of a parachutist (mass $m = 100kg$) if his main parachute and the secondary one fail to be deployed.

The average coefficient of resistance of the parachutist falling with closed parachute is $c = 0.8$ while the average cross section area can be considered $A = 0.5m^2$. The density of air on the ground is $d = 1.205 kg/m^3$.

Express the ground speed in:
(b) Kilometers per hour;
(c) Miles per hour.

See the answer in Appendix B

Exercise 1.4 Another Form of the Speed Formula

(a) Write the formula (1.8) considering that the cross-section area of the hemisphere parachute is

$$A = \pi \cdot D^2 / 4 \tag{1.10}$$

(b) Use the obtained formula to find the ground speed of the parachutist if the diameter of the parachute is $d = 7m$.

Consider the following data: $m = 100kg$, $d = 1.205kg/m^3$, $c = 1.33$, $g = 9.81m/s^2$.

Solution (a)

Substituting (1.10) in (1.8), we find the speed of the parachutist:

$$v = \sqrt{\frac{8mg}{\pi \cdot c \cdot d \cdot D^2}}\ . \tag{1.11}$$

Answer (b) see appendix B

Simplified Formula for the Ground Speed of the Falling Object

Consider the formulas (1.8) and (1.11), that can be used respectively to estimate the ground speed of a falling object and the ground speed of an object (parachutist falling with an open hemisphere parachute) whose effective cross-sectional area is equal to the area of the circle, i.e.

$$v = \sqrt{\frac{2mg}{cdA}}\ , \tag{1.12}$$

and

$$v = \sqrt{\frac{8mg}{\pi \cdot c \cdot d \cdot D^2}}\ . \tag{1.13}$$

The density of air on the ground is $d = 1.205kg/m^3$.

Substituting $d = 1.205kg/m^3$, $\pi = 3.1416$ and $g = 9.81m/s^2$ in formula (1.12) and (1.13) we obtain a simplified formulas respectively for the ground speed of the falling object:

$$v = 4.035\sqrt{\frac{m}{cA}}\ . \tag{1.14}$$

and for the falling speed of a parachutist falling with deployed parachute:

$$v = 4.553\sqrt{\frac{m}{c \cdot D^2}}\ . \tag{1.15}$$

Exercise 1.5 The Ground Speed of the Raindrop

The raindrops in absence of the ground wind, falls with a constant speed. Find the ground speed of a raindrop with mass $m = 1.06 \times 10^{-5} kg$, diameter $D = 3mm = 0.003m$ considering that the raindrop has a spherical form.

From table 1, appendix A, we find that the drag coefficient of a spherical raindrop is $c = 0.48$. The ground speed of the raindrop can be estimated using formula (1.15), i.e.

$$v = 4.553\sqrt{\frac{m}{c \cdot D^2}}$$

See answer in Appendix B.

Example 1.6 The Diameter of the Parachute

The U.S.A Red-Cross wants to distribute a mass of goods, assembled in a container of mass $m = 500kg$, to a population isolated at a remote disastrous region. A military helicopter has got the task to distribute the goods.

Use the formula (1.15) to find the diameter D of the hemisphere parachute needed to launch the container so that the ground speed of the container will not be greater than $v = 3m/s$.

Solution

Substituting in formula (1.15) i.e. in:

$$v = 4.553\sqrt{\frac{m}{c \cdot D^2}} \quad , \qquad (1.16)$$

we have:

$$3 = 4.553\sqrt{\frac{(500)}{(1.33)D^2}} \; .$$

Solving the above equation we find that the diameter of the parachute must be not less than:

$$D = 29.43m.$$

Note: The container must be strong enough to resist the impact on the ground with speed $v = 3m/s$.

Think:

(a) How to test that the container can resist the impact? (See exercise 1.8).

(b) In case that the test shows that the container can not resist the impact on the ground what we should do to avoid the destruction of the goods? (Answer: Use a parachute with a larger diameter to decrease the impact speed of the container.)

Example 1.7 Experimental Estimation of the Drag Coefficient

A student constructed a small hemisphere parachute with diameter $D = 0.6m$. To estimate the drag coefficient of the parachute he attached to the parachute a small object, such that the total mass of the parachute and the object was $m = 0.250kg$. Then he launched the parachute from the terrace of a high building.

After falling down (around 10 meters), the student started measuring the time of flight when the parachute was at the altitude $h = 20m$ over the ground. The measured time was $t = 6.2s$.

(a) Use (1.15) to find a formula for the drag coefficient c.
(b) Use the obtained formula to find the drag coefficient of the student's parachute.

Solution

(a) Solving for c the equation (1.15), i.e the equation

$$v = 4.553\sqrt{\frac{m}{c \cdot D^2}},$$

we obtain the following formula that can be used to estimate the drag coefficient of a hemisphere parachute:

$$c = \frac{20.73 \cdot m}{D^2 v^2} \qquad (1.17)$$

(b) The falling speed of the parachute is

$$v = \frac{h}{t} = \frac{20}{6.20} = 3.226m/s.$$

Employing (1.17) we find that the drag coefficient of the student's parachute is

$$c = \frac{20.73 \cdot m}{D^2 v^2} = \frac{20.73 \cdot (0.250)}{(0.6)^2 (3.226)^2} = 1.38.$$

Exercise 1.8 Training (Testing) Tower

The speed v of a body falling from an altitude h, in absence of air, is given by the known formula of the free fall:

$$v^2 = 2gh. \qquad (1.18)$$

a. Find the formula for the speed of a parachutist launched (with a non deployed parachute) from a training tower with height h.

b. What is the ground speed of the parachutist if the altitude of the tower is $h = 2m$?

c. Find the launching altitude h of the tower needed for the parachutist to reach the ground speed v.

d. What is the launching altitude h of the parachutist needed for him to reach a ground speed $v = 4.93 m/s$ (see Example 1.2)?

e. What will be the launching height of the container in order that the container reaches the impact speed $v = 3m/s$, if the container is launched from the testing tower (see exercise 1.6)?

Note: (1) For relatively small velocities the resistance of air can be neglected and the fall of an object or a parachutist (falling with closed parachute) can be considered as a free fall.

See answer in Appendix B.

Example 1.9 “Loosing” Weight (Optional)

The average force F that is exerted on the parachutist of mass m when its speed changes from v_1 to v_2 is given by the formula:

$$F = \frac{mv_2 - mv_1}{t} + mg \tag{1.19}$$

where mg is the weight P of the parachutist ($P = mg$) while g is the gravity acceleration: $g = 9.81 m/s^2$. The time of flight is denoted by t.

The falling parachutist (skydiver) deploys the parachute at the altitude 800 when his/her speed is 50m/s.

Find the average force that is exerted on the parachutist during the deployment considering that the interval of time during which the parachute is fully deployed is $t = 3s$ and that the speed of parachutist at the end of the interval is 10m/s.

The mass of the parachutist/skydiver including his parachute and other equipments is 100 kilogram.

Solution

Substituting in (2.1) we find that the average force of resistance that is exerted on the parachutist/skydiver is

$$F = \frac{mv_2 - mv_1}{t} + mg = \frac{m(v_2 - v_1)}{t} + mg = \frac{100 \cdot (10 - 50)}{3} + 100 \cdot (9.81) = -352.331N$$

We can use the Newton’s second law $F = ma$ to find the average acceleration of the parachutist during the parachute deployment.

The acceleration of the parachutist is

$$a = \frac{F}{m} = \frac{-352.331}{100} = -3.52 m/s^2 .$$

Note that the negative sign shows that the force and the acceleration are directed opposite to the direction of fall.

During the parachute deployment, the parachutist feels a lifting force and that he is loosing weight being “lifted up” (flying) in the opposite direction of fall.

Fall of Objects in Presence of Wind. Using the Pythagorean Theorem

The fall of the parachutist in presence of the horizontal wind is affected by the force that the wind exerts horizontally on the falling object. Immediately after the object is launched vertically in a windy weather, it experience action of wind. The magnitude of the force F_w exerted horizontally by the wind on a vertically falling object (parachutist) is given by the formula:

$$F_w = \frac{c}{2} dA(w - v_x)^2 \tag{1.20}$$

where:

- d is the density of air. On the ground the air density is $d = 1.205 kg/m^3$
- w is the speed of the horizontal wind.
- A is the largest effective cross-sectional area of the object that is perpendicular to the wind direction.
- c is the drag coefficient of the object (called also the coefficient of resistance).
- v_x is the component of speed in the horizontal direction.
- $w - v_x$ is the relative speed of the object, when $v_x < w$.

It can be shown that after a certain interval of time (i.e. after moving horizontally a certain distance) the horizontal speed of the falling object (parachutist) can be considered equal to the speed of wind w.

The motion of a falling object (parachutist) when the object approaches the ground can be considered as composed by the vertical fall with speed v determined by formula (1.3) and the horizontal motion with speed equal to the wind speed w.

Employing the Pythagorean Theorem, we find that the ground speed of the falling object (parachutist) in presence of air is

$$u = \sqrt{v_y^2 + w^2} \tag{1.21}$$

Example 1.10 The Speed of the Parachutist in Presence of Wind

A parachutist with mass $m = 90kg$ falls on the ground in presence of a wind that blows with a speed $w = 3m/s$. The parachute is hemisphere with diameter $D = 7.5m$.

(a) Find the ground speed of the parachutist.

(b) What will be the height of the training tower platform that assures the parachutist will fall on the ground with the ground speed found in (a).

Solution

(a) The vertical speed of the parachutist is

$$v = 4.553\sqrt{\frac{m}{c \cdot D^2}} = 4.553\sqrt{\frac{90}{1.33(7.5)^2}} = 4.99 m/s$$

Using formula (1.21) we find that the ground speed of the parachutist is

$$u = \sqrt{v^2 + w^2} = \sqrt{(4.99)^2 + 3^2} = 5.83 m/s\,.$$

(b) The height of the platform of the training tower (see exercise 1.8) is

$$h = u^2 / 2g = (5.83)^2 /(2 \cdot 9.81) = 1.73m$$

Example 1.10 The Ground Speed of a Table Tennis Ball

A table tennis ball is launched from an altitude $h = 30m$ (a multistory building) in presence of wind that blows with speed $w = 3.2m/s$. Consider that when the ball falls near the ground the vertical sped of the ball can be calculated using (1.15), while the horizontal speed becomes approximately equal to the wind speed.

(a) Find the ground vertical speed of the ball.

(b) Find the resultant ground speed of the ball.

A table tennis ball that is used in international competition has a diameter $D = 40mm = 0.04m$ and a mass of $m = 2.7 gram = 0.0027kg$.

Solution

From table in appendix A we find that the drag coefficient of the ball is $c = 0.48$.

(a) The ground vertical speed of the ball is

$$v = 4.553\sqrt{\frac{m}{c \cdot D^2}} = 4.553\sqrt{\frac{0.0027}{0.48(0.04)^2}} = 8.54 m/s\,.$$

(b) The ground speed (resultant speed) of the table tennis ball is

$$u = \sqrt{v^2 + w^2} = \sqrt{(8.54)^2 + (3.2)^2} = 9.12 m/s\,.$$

Exercise 1.11 Ground Speed of a Hailstone

The hailstone can damage (people, harvest, trees, etc.) if the diameter is over one inch (2.54cm), i.e. $D = 0.0254$ meters. The hailstone is considered as a sphere.

(a) Find the threshold ground speed of the hailstone considering that the mass of such hailstone is $m = 8$ gram.

(b) What is the ground speed of the hailstone if there is a wind with speed $w = 36$ kilometers per hour?

Answer in Appendix B

New Formulae for the Ground Speed of a Hailstone

Using formula (1.13), it can be shown that the ground speed of a hailstone (considered spherical) can be estimated by the formula:

$$v = 3.136\sqrt{\frac{d_1 D}{d}}, \quad (1.22)$$

where

- d is the density of air; $d = 1.205 kg/m^3$
- d_1 is the density of hailstone , $d_1 = 916.7 kg/m^3$.
- D is the diameter of the hailstone.

Substituting the given densities we find another simple formula for the ground speed of the hailstone:

$$v = 86.50\sqrt{D} \quad (1.23)$$

Note: To estimate the ground speed of a hailstone using formula (1.23) we need to measure only the average diameter of the hailstone.

Exercise 1.12

On July 23, 2010, after a storm, a resident in S. Dakota found a hailstone with (measured) diameter 8 inches with a mass 1 pound and 15 ounces. Estimate the ground speed of the hailstone?

Answer in Appendix B

Assignment 1 (Optional)

Using the formula (1.13), derive formula (1.23) considering that the mass of a spherical object can be estimated using the following formula:

$$m = \frac{4}{3}\pi \frac{D^3}{8} \cdot d_1 \quad (1.24)$$

Assignment 2 (Optional)

Explain the Mathematical Basis of “Simple Wind Monitor” presented in internet in page 28 at:
http://www.uwsp.edu/cnr/wcee/keep/Resources/Agriculture/Activitiespdfs/Ag%20Ed%20Measuring%20Wind%20Speed.pdf

The wind monitor is constructed with a fishing line, a tennis table ball, and a protractor. U can get help from your physics instructor or your Math1275 instructor.

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