Sec 2.1 : Linear FO Equations - Method of Integrating Factors

Standard forms of linear FO equations :

(1) $P(t) \cdot \frac{dy}{dt} + Q(t) \cdot y = G(t)$

(2) $\frac{dy}{dt} + p(t) \cdot y = g(t)$ ← Proceed from this form for method of integrating factors

Example : Given FO equation in form (1) : $t^2 \cdot \frac{dy}{dt} + 2ty = \cos t$
convert to form (2) by dividing through : $\frac{dy}{dt} + \frac{2t}{t^2} y = \frac{\cos t}{t^2}$

Integrating factor : Given eqn in form (2), multiply equation through by

$$\boxed{\mu(t) = \exp\left(\int p(t)\, dt\right)}$$

(where $\int p(t)\, dt$ is the antiderivative of $p(t)$.)

Then the LHS of the diff eqn becomes the result of a product rule :

$$\underbrace{\overbrace{\exp\left(\int p(t)\, dt\right)}^{\mu(t)} \cdot \frac{dy}{dt} + \overbrace{\exp\left(\int p(t) dt\right)}^{\mu(t)} \cdot p(t) \cdot y}_{\frac{d}{dt}\left[\exp\left(\int p(t) dt\right) y\right]} = \overbrace{\exp\left(\int p(t)\right)}^{\mu(t)} \cdot g(t)$$

(Product rule : $\frac{d}{dt}\left(\exp\left(\int p(t)\, dt\right) \cdot y\right) = \exp\left(\int p(t)\, dt\right) \cdot \frac{dy}{dt} + \underbrace{\frac{d}{dt}\left(\exp\left(\int p(t)\, dt\right)\right)}_{\exp\left(\int p(t)\, dt\right) \cdot p(t)} \cdot y$ } by chain rule

So we have $\frac{d}{dt}\left[\exp\left(\int p(t)\, dt\right) \cdot y\right] = \exp\left(\int p(t) dt\right) \cdot g(t)$

Integrate both sides :

$$\boxed{\underbrace{\exp\left(\int p(t)\, dt\right)}_{\mu(t)} \cdot y = \int \underbrace{\overbrace{\exp\left(\int p(t)\, dt\right)}^{\mu(t)} \cdot g(t)\, dt}}$$

So to finish solving for y, you need to integrate $\mu(t) \cdot g(t)$.

Example : $\frac{dy}{dt} + \overbrace{\left(\frac{2}{t}\right)}^{p(t)} y = \frac{\cos t}{t^2}$ $\Rightarrow \mu(t) = \exp\left(\int \frac{2}{t}\, dt\right) = \underbrace{\exp(2 \ln|t|)}_{\text{since } e^{2\ln|t|} = \left(\underbrace{e^{\ln|t|}}_{t}\right)^2} = \underline{\underline{t^2}}$

$\Rightarrow t^2 \cdot \frac{dy}{dt} + \frac{2}{t} \cdot t^2 \cdot y = \frac{\cos t}{t^2} \cdot t^2$

$\underbrace{t^2 \cdot \frac{dy}{dt} + 2t \cdot y}_{\frac{d}{dt}(t^2 \cdot y)} = \cos t$

$\frac{d}{dt}(t^2 \cdot y) = \cos t$

$\rightarrow t^2 \cdot y = \int \cos t\, dt$

$\rightarrow t^2 \cdot y = \sin t + C$

$$\boxed{y = \frac{1}{t^2}(\sin t + C)}$$

Summary :

(1) Put given linear FO eqn in form $\frac{dy}{dt} + p(t) \cdot y = g(t)$

(2) Find antiderivative of $p(t)$, i.e., $\int p(t)\, dt$

(3) Multiply both sides of eqn by integrating factor : $\mu(t) = \exp\left(\int p(t)\, dt\right)$

(4) Verify that LHS is $\frac{d}{dt}(\mu(t) \cdot y)$

(5) Find antiderivative of RHS, i.e. find $\int \mu(t) \cdot g(t)\, dt$.