1. Solve the following system of equations. Find **the row echeleon form** and the **row-reduced echeleon form** using elementary row operations and then solve it. **Do not use calculator** to solve this problem. You can check your solution using calculator.

   (a) $w - x - y + 2z = 1,\ 2w - 2x - y + 3z = 3,\ -w + x - y = -3$
   (b) $x + y + z = 3,\ x + 2y + 2z = 5,\ 3x + 4y + 4z = 12$

2. Determine whether the system is consistent, inconsistent, or dependent.
   (a) $3x + 2y = 15,\ 6x + 4y = 30$
   (b) $3x + 2y - 5z = 4,\ x + y - 2z = 1,\ 5x + 3y + 8z = 6$

3. Find the unique value of t for which the following system has a solution. Determine the basic and free variable, if applicable.
   - $x_1 + x_3 - x_4 = 3,\ 2x_1 + 2x_2 - x_3 - 7x_4 = 1,\ 4x_1 - x_2 - 9x_3 - 5x_4 = t,\ 3x_1 - x_2 - 8x_3 - 6x_4 = 1.$

4. Show that the following set spans R^3 and then write the vector **b** as a linear combination of these vectors.
   $\{[1\ {-1}\ 0]^T, [1\ 1\ 1]^T, [1\ 0\ 1]^T\}$; $\mathbf{b} = [-1\ 2\ 2]^T$

5.

Let $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$. It can be shown that $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Use this fact to exhibit $\mathbf{b}$ as a specific linear combination of the columns of A.

6. Let $v_1 = (-2, 0, 1)$, $v_2 = (1, -1, 2)$ and $v_3 = (4, -2, 3)$. Are these vectors linearly dependent or independent? Justify your answer.

7. Go through the assignment problems on Concept Checking/True and False – Note just by writing true or false will not give you any point. You need to justify your conclusion i.e. you need to provide explanation why you think the following statement is true or false.

Soln. MAT2580 Exam 1 Review Soln.

1.
$$\begin{aligned} w - x - y + 2z &= 1 \\ 2w - 2x - y + 3z &= 3 \\ -w + x - y &= -3 \end{aligned}$$

Note: We've 3 equ. but 4 unknowns.

$$\begin{pmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{pmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{pmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{R_3+R_2} \begin{pmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

row echleen form

reduced row echlen form

The soln. $y - z = 1 \Rightarrow \boxed{y = z}$

$w = x + y - 2z + 1$ $\qquad$ y, w basic var.

$\Rightarrow w = x + z - 2z + 1$ $\qquad$ x, z free var.

$\Rightarrow \boxed{w = x - z + 1}$

soln. $\vec{x} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z + 1 \\ x \\ z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} x + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

1(b) $x+y+z=3$

$x+2y+2z=5$

$3x+4y+4z=12$

$$A\,|\,\vec{b}$$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 12 \end{pmatrix} \xrightarrow[R_3-3R_1]{R_2-R_1} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow$$ The system is inconsistent!

$$\xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2-2R_3} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1-R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ Reduced row echleon form.

$$\boxed{\text{No soln.}}$$

$3x + 2y = 15, \quad 6x + 4y = 30$

#2. a $\quad \overset{A|\vec{b}}{\begin{pmatrix} 3 & 2 & 15 \\ 6 & 4 & 30 \end{pmatrix}} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 3 & 2 & 15 \\ 0 & 0 & 0 \end{pmatrix}$

~~2nd~~ $R_2 = 2R_1$

$3x + 2y = 15 \Rightarrow 3x = 15 - 2y$

$\boxed{x = 5 - \frac{2}{3}y}$

x basic var.
y free var.

System is consistent and has infinite # of soln.

#2(b) $\quad 3x + 2y - 5z = 4$
$\quad x + y - 2z = 1$
$\quad 5x + 3y + 8z = 6$

$$\begin{pmatrix} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & 8 & 6 \end{pmatrix}$$

$$\xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 1.9375 \\ 0 & 1 & 0 & -1.0625 \\ 0 & 0 & 1 & -0.625 \end{pmatrix}$$

$$\boxed{\begin{aligned} x &= 1.9375 \\ y &= -1.0625 \\ z &= -0.625 \end{aligned}}$$

#3

$$
\begin{aligned}
-x_1 + x_3 - x_4 &= 3\\
2x_1 + 2x_2 - x_3 - 7x_4 &= 1\\
4x_1 - x_2 - 9x_3 - 5x_4 &= t\\
3x_1 - x_2 - 8x_3 - 6x_4 &= 1
\end{aligned}
$$

$A \mid \vec{b}$

$$
\begin{pmatrix}
-1 & 0 & 1 & -1 & \vdots & 3\\
2 & 2 & -1 & -7 & \vdots & 1\\
4 & -1 & -9 & -5 & \vdots & t\\
3 & -1 & -8 & -6 & \vdots & 1
\end{pmatrix}
$$

$$
\xrightarrow{(-1)R_1}
\begin{pmatrix}
1 & 0 & -1 & 1 & -3\\
2 & 2 & -1 & -7 & 1\\
4 & -1 & -9 & -5 & t\\
3 & -1 & -8 & -6 & 1
\end{pmatrix}
\xrightarrow[\substack{R_3-4R_1\\ R_4-3R_1}]{R_2-2R_1}
\begin{pmatrix}
1 & 0 & -1 & 1 & -3\\
0 & 2 & 1 & -9 & 7\\
0 & -1 & -5 & -9 & 12+t\\
0 & -1 & -5 & -9 & +10
\end{pmatrix}
$$

$$
\xrightarrow{R_3-R_4}
\begin{pmatrix}
1 & 0 & -1 & 1 & -3\\
0 & 2 & 1 & -9 & 7\\
0 & 0 & 0 & 0 & 2+t\\
0 & -1 & -5 & -9 & 10
\end{pmatrix}
$$

To be consistent $t+2=0 \Rightarrow \boxed{t=-2}$

For $t=2$
$$
\begin{pmatrix}
1 & 0 & -1 & 1 & -3\\
0 & 2 & 1 & -9 & 7\\
0 & 0 & 0 & 0 & 0\\
0 & -1 & -5 & -9 & 10
\end{pmatrix}
$$

With $t = -2$ using rref in the calculator, we get,

$$\begin{pmatrix} 1 & 0 & 0 & 4 & -6 \\ 0 & 1 & 0 & -6 & 5 \\ 0 & 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_3 + 3x_4 = -3 \Rightarrow \boxed{x_3 = -3 - 3x_4}$$
$$x_2 - 6x_4 = 5 \Rightarrow \boxed{x_2 = 5 + 6x_4}$$
$$x_1 + 3x_4 = -6 \Rightarrow \boxed{x_1 = -6 - 3x_4}$$

x_1, x_2, x_3 basic var,
x_4 free var.

\# 4 Are these indept.?

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Note: "T" means transpose.

$$(1 \ -1 \ 0)^T = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Solving $A\vec{x} = 0$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{2R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

ref form

The only soln.

$x_1 = x_2 = x_3 = 0$ or $c_1 = c_2 = c_3 = 0$

$\Rightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are indept. & it should span $\mathbb{R}^3$, since each
vector has 3 elem

there are 3 ~~[illegible]~~ indept. vectors.

$$\vec{b} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$A|\vec{b}$

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$\Rightarrow$ $c_1 = -3$, $c_2 = -1$, $c_3 = 3$.

$$\vec{b} = -3\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

#5. Given $\vec{p}$ is a soln. of $A\vec{x} = \vec{b}$.

$\Rightarrow$ $A\vec{p} = \vec{b}$

Therefore, the elements of $\vec{p}$ ~~provide~~ are actually the coeff. i.e

$$3\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 5 \\ 1 \\ -8 \end{pmatrix} + 0\begin{pmatrix} -2 \\ 9 \\ -1 \end{pmatrix} - 4\begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix} = \begin{pmatrix} -7 \\ 9 \\ 0 \end{pmatrix}$$

Check this!

#6 $\vec{v}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly dependent since we've nontrivial soln. for c_1, c_2, c_3.

$$c_2 + 2c_3 = 0 \Rightarrow c_2 = -2c_3$$

$$c_1 - c_3 = 0 \Rightarrow c_1 = c_3$$

We can choose any non-zero value of c_3 and we'll end up with non-zero values for c_1 & c_2.