Soln. Review Exam #2

#11 Given A, B, X are $n \times n$ matrices.

A, X, and $A - AX$ invertible.

Given $(A - AX)^{-1} = X^{-1}B$.

Note: If $AC = I$ ~~and~~ or $DA = I$ then A is invertible by the theorem where A is a square matrix

(a) $\underbrace{(A-AX)(A-AX)^{-1}}_{I} = (A-AX)X^{-1}B$ (Left multiply by $A-AX$)

$\Rightarrow I = (AX^{-1} - A(XX^{-1}))B$

$\Rightarrow I = (AX^{-1} - A)B$ (since X is invertible $XX^{-1} = I$)

$\Rightarrow (AX^{-1} - A)B = I$

$\Rightarrow$ By thrm 8. (p-112) - (j) B ~~has an~~ is invertible & $\boxed{B^{-1} = (AX^{-1} - A)}$.

(b) $(A - AX)^{-1} = X^{-1}B$

We need to solve for X.

From part (a) $(AX^{-1} - A)B = I$

$\Rightarrow (AX^{-1} - A)\underbrace{BB^{-1}}_{I} = \underbrace{IB^{-1}}_{B^{-1}}$ (We know now B^{-1} exists. so right multiply by B^{-1})

$\Rightarrow AX^{-1} - A = B^{-1}$

$\Rightarrow AX^{-1} = A + B^{-1}$

$\Rightarrow \underbrace{A^{-1}(A}_{I}X^{-1}) = A^{-1}(A + B^{-1})$ (Right multiply by A^{-1}. It is given that A^{-1} exists)

$\Rightarrow X^{-1} = A^{-1}(A + B^{-1})$

Now left multiply by X $\Rightarrow \underbrace{XX^{-1}}_{I} = XA^{-1}(A + B^{-1})$

$\Rightarrow XA^{-1}(A + B^{-1}) = I \Rightarrow$

We know X is ~~invertible~~ $n \times n$ matrix
& Since A and B are $n \times n$
$A^{-1}(A+B^{-1})$ is an $n \times n$ matrix.

Now $\underbrace{X}_{n\times n}\underbrace{\left[A^{-1}(A+B^{-1})\right]}_{n\times n} = I$

$\Rightarrow$ by thrm. 8 (p-112) (j) $A^{-1}(A+B^{-1})$ is also invertible and that equals to X^{-1}.

$\Rightarrow X^{-1} = A^{-1}(A+B^{-1})$

$X^{-1} = \left((A+B^{-1})^{-1}A\right)^{-1}$ $\longrightarrow$ Note: $(AB)^{-1} = B^{-1}A^{-1}$ & $(A^{-1})^{-1} = A$

$\Rightarrow \boxed{X = (A+B^{-1})^{-1}A}$

$\Rightarrow \left((A+B^{-1})^{-1}A\right)^{-1}$
$= A^{-1}\left((A+B^{-1})^{-1}\right)^{-1}$
$= \boxed{A^{-1}(A+B^{-1})}$

#12. $A = \begin{pmatrix} 5 & 3 & 2 & -6 & -8 \\ 4 & 1 & 3 & -8 & -7 \\ 5 & 1 & 4 & 5 & 19 \\ -7 & -5 & -2 & 8 & 5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

pivot columns.

Therefore the bases for the column space
$= \left\{ \begin{pmatrix} 5 \\ 4 \\ 5 \\ -7 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \\ -5 \end{pmatrix}, \begin{pmatrix} -6 \\ -8 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} -8 \\ -7 \\ 19 \\ 5 \end{pmatrix} \right\}$ Ans.