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HW # 2 – pg 21: 1-21 (odd)

1. Determine which matrices are in reduced echelon form and which others are only in echelon form.

a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ Reduced Row Echelon Form; See definition below

b. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Reduced Row Echelon Form; See definition below

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Not in Echelon form; Doesn't match criteria 1

d. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ Row Echelon Form; Doesn't match criteria 4

Definition:

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or r**educed row echelon form**):

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

3. Row reduce the matrices to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

$$\underset{E_{Replace}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}} * \underset{\text{Original Matrix}}{\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}} = \underset{\text{Product}_1}{\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2-2 & 4-4 & 6-8 & 8-16 \\ 3-3 & 6-6 & 9-12 & 12-24 \end{bmatrix}}$$

$$\underset{E_{Replace}}{\begin{bmatrix} 1 & 2 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}} * \underset{\text{Product}_1}{\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -3 & -12 \end{bmatrix}} = \underset{\text{Product}_2}{\begin{bmatrix} 1 & 2 & 4-4 & 8-16 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -3+3 & -12+12 \end{bmatrix}} =$$

Reduced
Echelon Form

$$\begin{bmatrix} 1 & 2 & 0 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first and third column are the pivot columns with the pivot positions dentified by the red circles.

5. Describe the possible echelon forms of a nonzero 2 * 2 matrix. Use the symbol ■, *, and 0, as in the first part of Example 1.

a) $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}$ - See definition above

b) $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}$ - See definition above

c) $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$ - See definition above

Find the general solutions of the systems whose augmented matrices are given in Exercises 7–14.

7. $\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & 7 \\ -3+3 & -9+9 & -7+12 & -6+21 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 5 & 15 \end{bmatrix} =$

$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 5 & 15 \end{bmatrix}$; $x_1+3x_2+4x_3=7$, x_2 is free, $5x_3=15$ ~ $\mathbf{x_1}=7-12-3$, $\mathbf{x_2}= -5-3x_2$, x_2 is free, $\mathbf{x_3}=3$

9. $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} * \begin{bmatrix}0 & 1 & -2 & 3\\ 1 & -3 & 4 & -6\end{bmatrix} = \begin{bmatrix}1 & -3 & 4 & -6\\ 0 & 1 & -2 & 3\end{bmatrix}$

$\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix} * \begin{bmatrix}1 & -3 & 4 & -6\\ 0 & 1 & -2 & 3\end{bmatrix} = \begin{bmatrix}1 & -3+3 & 4-6 & -6+9\\ 0 & 1 & -2 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & -2 & -3\\ 0 & 1 & -2 & 3\end{bmatrix}$

x_1-2x_3=3, x_2-2x_3=3, x_3 is free ~ $\mathbf{x_1}$=$\mathbf{x_2}$=3+2x_3, $\mathbf{x_3}$ is free

11. $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -3\\ -2 & 0 & 1\end{bmatrix} * \begin{bmatrix}3 & -2 & 4 & 0\\ 9 & -6 & 12 & 0\\ 6 & -4 & 8 & 0\end{bmatrix} = \begin{bmatrix}3 & -2 & 4 & 0\\ 9-9 & -6+6 & 12-12 & 0\\ 6-6 & -4+4 & 8-8 & 0\end{bmatrix} = \begin{bmatrix}3 & -2 & 4 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

$\mathbf{x_1}$=1/3(2x_2-4x3), $\mathbf{x_2}$ is free, $\mathbf{x_3}$ is free

13. $\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} * \begin{bmatrix}1 & -3 & 0 & -1 & 0 & -2\\ 0 & 1 & 0 & 0 & -4 & 1\\ 0 & 0 & 0 & 1 & 9 & 4\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & -3 & 0 & -1+1 & 9 & -2+4\\ 0 & 1 & 0 & 0 & -4 & 1\\ 0 & 0 & 0 & 1 & 9 & 4\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} =$

$\begin{bmatrix}1 & -3 & 0 & 0 & 9 & 2\\ 0 & 1 & 0 & 0 & -4 & 1\\ 0 & 0 & 0 & 1 & 9 & 4\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$; x_3 is free, x_5 is free, x_1=2+3x_2-9x_5, x_2=1+4x_5,x_4=4-9x_5, x_5 is free ~

$\mathbf{x_1}$=5+3x_5, $\mathbf{x_2}$=1+4x_5, $\mathbf{x_3}$ is free, $\mathbf{x_4}$=4-9x_5, $\mathbf{x_5}$ is free

15. Use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

a. $\begin{bmatrix}\blacksquare & * & * & *\\ 0 & \blacksquare & * & *\\ 0 & 0 & 0 & 0\end{bmatrix}$ Consistent with many solutions; see definition above

b. $\begin{bmatrix}0 & \blacksquare & * & * & *\\ 0 & 0 & \blacksquare & * & *\\ 0 & 0 & 0 & \blacksquare & 1\end{bmatrix}$ Consistent with many solutions; see definition above

17. Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} * \begin{bmatrix} 1 & -1 & 4 \\ -2 & 3 & h \end{bmatrix} = \begin{bmatrix} 1 & -1 & 4 \\ -2+2 & 3-2 & h+8 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & h+8 \end{bmatrix}$$

$x_1=4+x_2$, $x_2=h+8$; $\mathbf{x_1}=12+h$ ~ All values of h will make this a consistent linear system.

19. Choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

$x_1 + hx_2 = 2, \quad 4x_1 + 8x_2 = k$

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} * \begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} = \begin{bmatrix} 1 & h & 2 \\ 4-4 & 8-4h & k-8 \end{bmatrix} = \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$$

a. Inconsistent when $h = 2$ & $k \neq 8$ - See below Theorem 2

b. Unique solution when $h \neq 2$ - See below Theorem 2 *ii*

c. Many solutions when $h = 2$ and $k = 8$ – See below Theorem 2 *i*

Theorem 2:
Existence and Uniqueness Theorem
A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \ \ \ldots \ \ 0 \ \ b] \text{ with b nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

21. True/False.

a. In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations. - **FALSE**

b. The row reduction algorithm applies only to augmented matrices for a linear system. - **FALSE**

c. A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix. - **TRUE**

d. Finding a parametric description of the solution set of a linear system is the same as solving the system. - **TRUE**

e. If one row in an echelon form of an augmented matrix is $[0 \quad 0 \quad 0 \quad 5 \quad 0]$, then the associated linear system is inconsistent.- **FALSE**